Lecture 17

Elimination of ε-productions

$A \to \varepsilon$ ... an ε-production

As ε-productions produce nothing, we want to remove them from the CFG G while preserving the language $L(G)$.

G ..... $L(G)$

$\Downarrow$ $\Downarrow$

G' $L(G') = L(G) - \{\varepsilon\}$

without ε-prod.

Observe: Let $A \Rightarrow^* \varepsilon$ be a derivation.

Then we have:

- $A \Rightarrow \varepsilon$ or
- $A \Rightarrow A_1 A_2 \cdots A_k \Rightarrow^* \varepsilon$

where $A_1 \Rightarrow^* \varepsilon$, $A_2 \Rightarrow^* \varepsilon$, ..., $A_k \Rightarrow^* \varepsilon$.

Definition: A nonterminal $A \in V$ is **nullable** if there is a derivation $A \Rightarrow^* \varepsilon$.

Example: G:

$S \to ACA$

$A \to aAa \mid B \mid C$

$B \to bB \mid b$

$C \to cC \mid \varepsilon$

$N_0 = \{C\}$

$N_1 = \{C, A\}$

$N_2 = \{C, A, S\}$

$N_3 = N_2$

$\Rightarrow$ C, A, S are nullable.

- Let's take a production rule

$$A \to \alpha_0 A_1 \alpha_1 A_2 \alpha_2$$

where
- A_1, A_2 are nullable.
- $\alpha_0, \alpha_1, \alpha_2$ don't contain nullable symbols. (Note: $\alpha_0, \alpha_1, \alpha_2 \in (V \cup \Sigma)^*$)

Replace $A \to \alpha_0 A_1 \alpha_1 A_2 \alpha_2$ with the following rules:

$$A \to \alpha_0 \alpha_1 \alpha_2 \mid \alpha_0 \alpha_1 A_2 \alpha_2 \mid \alpha_0 A_1 \alpha_1 \alpha_2 \mid \alpha_0 A_1 \alpha_1 A_2 \alpha_2$$

After applying this transformation eagerly, remove all of the ε-productions.

Exercise: How to replace $A \to \alpha_0 A_1 \alpha_1 A_2 \alpha_2 A_3 \alpha_3$?

Example: G: $S \to ACA$
$A \to aAa \mid B \mid C$
$B \to bB \mid b$
$C \to cC \mid \varepsilon$

$N_0 = \{C\}$
$N_1 = \{C, A\}$
$N_2 = \{C, A, S\}$
$N_3 = N_2$

$\Rightarrow$ C, A, S are nullable.

G': $S \to ACA \mid CA \mid AA \mid AC \mid A \mid C$
$A \to aAa \mid aa \mid B \mid C$
$B \to bB \mid b$
$C \to cC \mid c$

Example: G: $S \to \underline{A}B\underline{C}B\underline{C}\underline{D}$
$A \to \underline{C}\underline{D}$
$B \to \underline{C}b$
$C \to \underline{A} \mid \varepsilon$
$D \to b\underline{D} \mid \varepsilon$

$N_0 = \{C, D\}$
$N_1 = \{C, D, A\}$
$N_2 = N_1$

$\Rightarrow$ C, D, A are nullable.

G': $S \to ABCBCD \mid BCBCD \mid ABBCD \mid ABCBD \mid ABCBC \mid$
$BBCD \mid BCBD \mid BCBC \mid ABBD \mid ABBC \mid ABCB$
$BBD \mid BBC \mid BCD \mid ABB \mid BB$
$A \to CD \mid C \mid D$

$B \to Cb \mid b$

$C \to A$

$D \to bD \mid b$

Note: $L(G') = L(G) - \{\varepsilon\}$

Elimination of Chain Productions

A production of the form $A \to B$, $A, B \in V$, is called a chain production.

e.g. $A \to B \Rightarrow C \Rightarrow D \Rightarrow E \Rightarrow F \Rightarrow \alpha$

s.t. $|\alpha| \geq 2$.

We want to replace this with $A \Rightarrow \alpha$.

Define, for each $A \in V$, the set

$CHAIN(A) = \{ B \in V \mid \exists$ a derivation $A \Rightarrow^* B$ using only chain rules. $\}$

Example:

$S \to A \mid cB$

$A \to C \mid D$

$B \to bB \mid b$

$C \to cC \mid c$

$D \to dD \mid d$

$CHAIN(S) = \{A, C, D\}$

$CHAIN(A) = \{C, D\}$

$CHAIN(B) = \emptyset$

$= CHAIN(C)$

$= CHAIN(D)$

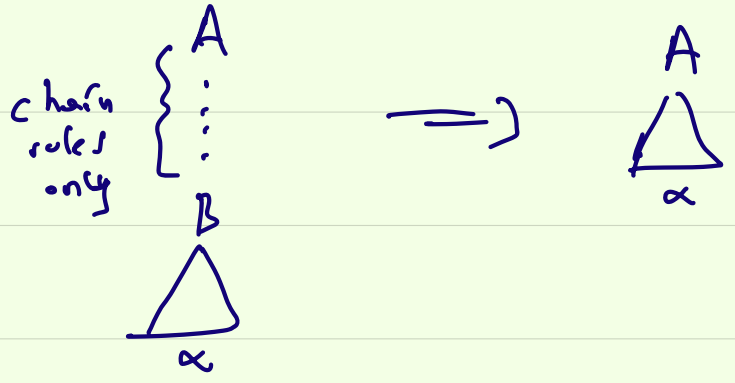
Q: How to remove chain rules?

A: Apply the following rule:

For every pair of nonterminals $A, B \in V$ s.t. $B \in CHAIN(A)$ and $B \to \alpha \in R$ (which is not a chain rule), add a rule $A \to \alpha$ to R.

Then remove all the chain rules.

i.e.



Example: G: $S \to A \mid cB$

$A \to C \mid D$

$B \to bB \mid b$

$C \to cC \mid c$

$D \to dD \mid d$

$CHAIN(S) = \{A, C, D\}$

$CHAIN(A) = \{C, D\}$

$CHAIN(B) = \emptyset$

$= CHAIN(C)$

$= CHAIN(D)$

G': $S \to cB \mid cC \mid c \mid dD \mid d$

$A \to cC \mid c \mid dD \mid d$

$B \to bB \mid b$

$C \to cC \mid c$

$D \to dD \mid d$

Example: G' : $S \to ACA \mid CA \mid AA \mid AC \mid A \mid C$

$A \to aAa \mid aa \mid B \mid C$

$B \to bB \mid b$

$C \to aC \mid c$

$CHAIN(S) = \{A, B, C\}$

$CHAIN(A) = \{B, C\}$

G'' : $S \to ACA \mid CA \mid AA \mid AC$

$aAa \mid aa \mid bB \mid b \mid cC \mid c$

$A \to aAa \mid aa \mid bB \mid b \mid cC \mid c$

$B \to bB \mid b$

$C \to aC \mid c$

Example:

G' : $S \to ABCBCD \mid BCBCD \mid ABBCD \mid ABCBD \mid ABCBC \mid$

$BBCD \mid BCBD \mid BCBC \mid ABBD \mid ABBC \mid ABCB$

$BBD \mid BBC \mid BCD \mid ABB \mid BB$

$A \to CD \mid C \mid D$

$B \to Cb \mid b$

$C \to A$

$D \to bD \mid b$

$CHAIN(S) = \emptyset$

$CHAIN(A) = \{C, D\}$

$CHAIN(C) = \{A, D\}$

$G'': S \rightarrow ABCBCD \mid BCBCD \mid ABBCD \mid ABCBD \mid ABCBC \mid$

$BBCD \mid BCBD \mid BCBC \mid ABBD \mid ABBC \mid ABCB$

$BBD \mid BBC \mid BCD \mid ABB \mid BB$

$A \rightarrow CD \mid bD \mid b$

$B \rightarrow Cb \mid b$

$C \rightarrow CD \mid bD \mid b$

$D \rightarrow bD \mid b$