Lecture 11

Example: Prove $L = \{ 0^{k^2} \mid k \geq 1 \}$ is not regular.

1) Let $n > 0$.

2) Let $z = 0^{n^2}$. Then $z \in L$ and $|z| \geq n$.

3) Consider a factorization $z = u \cdot v \cdot w$ s.t.

$|u \cdot v| \leq n$ and $|v| \geq 1$.

4) Let $i = 2$ and $z' = u \cdot v^i \cdot w = u \cdot v^2 \cdot w$

and $|v| = \ell$.

$\Rightarrow |z'| = |u \cdot v^2 \cdot w| = |u \cdot v \cdot w| + |v| = n^2 + \ell$

$n^2 < n^2 + \ell \leq n^2 + n < n^2 + 2n + 1 = (n+1)^2$

$\Rightarrow n^2 < n^2 + \ell < (n+1)^2$

$\Rightarrow n^2 + \ell$ is not a perfect square.

$\Rightarrow z' \notin L$.

5) L is not regular.

Exercise: In step 4, try $i = 0$.

Example: Prove $L = \{ ww \mid w \in \{0,1\}^* \}$ is not regular.

1) Let $n > 0$.

2) Let $z = 0^n 1 0^n 1$. Then $z \in L$ and $|z| \geq n$.

3) Consider a factorization $z = u \cdot v \cdot w$ s.t.
$|u \cdot v| \leq n$ and $|v| \geq 1$.

4) Let $i = 2$. Then $z' = u \cdot v^2 \cdot w =$
If $|v| = \ell$, then $z' = 0^{n+\ell} 1 0^n 1$.
Note: $\ell \geq 1$.
$\Rightarrow n + \ell \neq n$.
$\Rightarrow z' \notin L$.

5) L is not regular.

Example: Show that $L = \{ 0^i 1^j \mid i > j \}$ is not regular.

1) Let $n > 0$.

2) Let $z = 0^{n+1} 1^n$. Then $z \in L$ and $|z| \geq n$.

3) Consider a factorization $z = u \cdot v \cdot w$ s.t.

$|u \cdot v| \leq n$ and $|v| \geq 1$.

4) Let $i = 0$ and $|v| = \ell$

$\Rightarrow z' = u \cdot v^0 \cdot w = u \cdot w = 0^{n+1-\ell} 1^n$.

Note that $n+1-\ell \leq n$ since we have $\ell \geq 1$.

$\Rightarrow$ For z', $|z|_0 > |z|_1$ is not true.

$\Rightarrow z' \notin L$.

5) L is not regular.

Closure Properties of Regular Languages

Proposition: Regular languages are closed under union, concatenation and Kleene star operation.

Proof: Recall the definition of regular expressions and the proof equivalence between NFAs and RE. The result follows by def.

Proposition: Regular languages are closed under complement. (i.e. if L is regular, then so is $\bar{L}$)

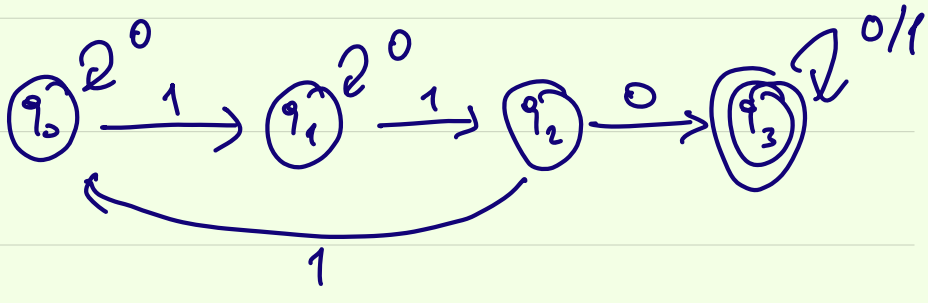
Proof: If M is a DFA for L, then we construct a DFA M' using M to accept $\bar{L}$ as follows:

$$M = (Q, \Sigma, \delta, q_0, F)$$

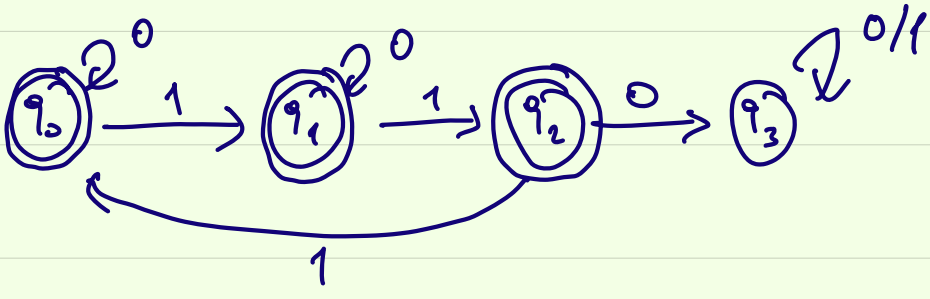
$$\Rightarrow M' = (Q, \Sigma, \delta, q_0, Q - F)$$

(i.e. make final states non final and non final states final in M')

e.g. M: L



M':



$\bar{L} = \Sigma^* - L$

Proposition Regular languages are closed under intersection. (i.e. if L_1, L_2 are reg., so is $L_1 \cap L_2$)

Proof: L_1, L_2 ... regular.

$$L_1 \cap L_2 = \overline{\overline{L_1 \cap L_2}}$$

$$= \overline{\overline{L_1} \cup \overline{L_2}}$$

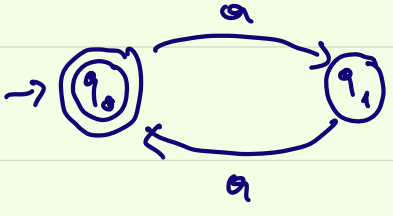
(Recall: $\overline{\overline{A}} = A$, $\overline{A \cap B} = \overline{A} \cup \overline{B}$)

We are done because $\overline{L_1}, \overline{L_2}$ are regular and reg. lang. are closed under union.

e.g. $L_1 = \{ a^{2n} \mid n \geq 0 \}$ $\quad \Sigma = \{a\}$

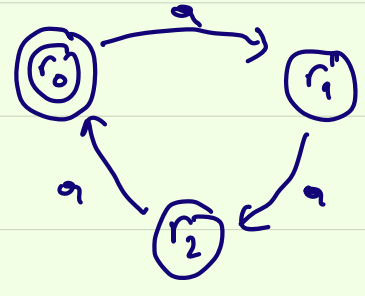
$L_2 = \{ a^{3n} \mid n \geq 0 \}$

M_1:



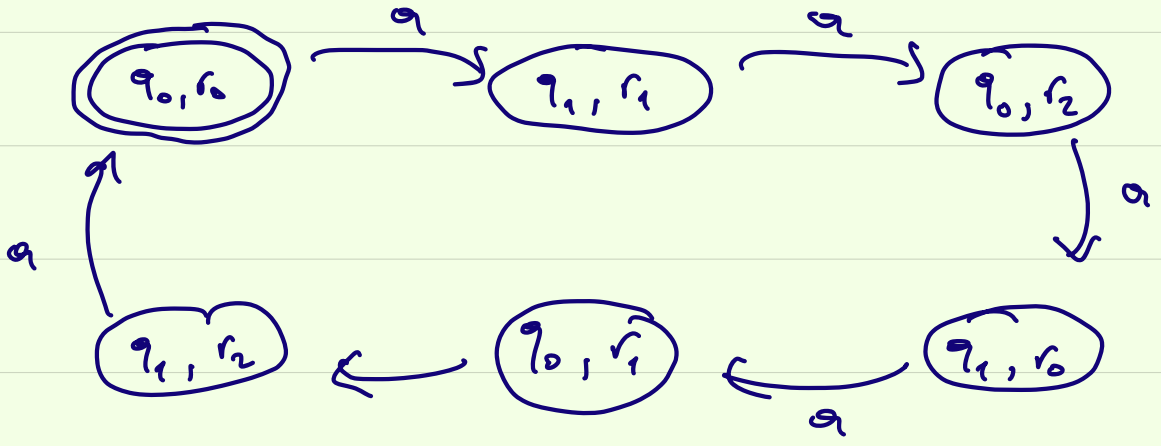
$Q_1 = \{q_0, q_1\}$

M_2:



$Q_2 = \{r_0, r_1, r_2\}$

M_3 for $L_1 \cap L_2$ ... $Q_3 = Q_1 \times Q_2$



$$L_3 = L_1 \cap L_2 = \{ a^{6n} \mid n \geq 0 \}$$

Example: Prove that $L = \{ 0^i 1^j \mid i \neq j \}$ is not regular

- We know that $L' = \{ 0^i 1^j \mid i = j \}$ is not regular.
- Suppose L is regular.

  $L \cap L(0^*1^*) = L'$ should be regular due to closure under complement and intersection. That's a contradiction.

  L' is known to be nonregular.

  $\Rightarrow$ L is not regular.

Example: Let $L \subseteq \Sigma^*$ regular lang. and $w \in \Sigma^*$

Prove that $L' = \{x \in L \mid x = \Sigma^* w \Sigma^*, |x| = 2k\}$

is regular.

Proof: $L_1 = \{\Sigma^* w \Sigma^* \mid w \in \Sigma^*\}$ ... regular

$L_2 = L((\Sigma^*)^2)$ .. regular

(i.e. $L((0+1)^* \cdot (0+1)^*)$

$\Rightarrow L' = L \cap L_1 \cap L_2$ regular due to closure under intersection

Exercise: If L_1, L_2 are regular, is $L_1 - L_2$ regular?