Let $M = (S, \mathcal{I})$ be a matroid.

The <u>rank function</u> $r_M : 2^S \to \mathbb{Z}_+$.

For $X \subseteq S$, $r_M(X)$ is the size of a max independent subset of X.

For any $X \subseteq S$, the <u>span</u> of X

$\text{span}_M(X) := \{y : y \in S, r_M(X + y) = r_M(X)\}$.

Set $X \subseteq S$ is <u>spanning</u> if $\text{span}_M(X) = S$.

Set $X \subseteq S$ is a <u>flat</u> of M iff $\text{span}_M(X) = X$.

A matroid $M = (S, \mathcal{I})$ is <u>connected</u> if $r_M(U) + r_M(S \setminus U) > r_M(S)$.

Equivalently, for each $s, t \in S$, there is a circuit containing both s & t.

Given a matroid $M = (S, \mathcal{I})$, its dual matroid $M^* = (S, \mathcal{I}^*)$ has

$$\mathcal{I}^* := \{I \subseteq S \mid S \setminus I \text{ is spanning in } M, \text{ i.e., } r_M(S \setminus I) = r_M(S)\}.$$

e.g. given connected $G = (V, E)$ with graphic matroid M. $M^* = (E, \mathcal{I}^*)$ has $I \in \mathcal{I}^*$ if $G - I$ is still connected.

Claim:
1) $M^{**} = M$
2) B is a base of M^* iff $S \setminus B$ is a base of M.

3) $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$

4) M is connected iff M^* is connected.

Given $M = (S, \mathcal{I})$ & $e \in S$, <u>deleting</u> e from M generates a new matroid

$$M' = M \setminus e = (S - e, \mathcal{I}'),$$

where $\mathcal{I}' = \{ I - e \mid I \in \mathcal{I} \}$, i.e. M restricted to $S - e$.

For $Z \subseteq S$, $M \setminus Z$ is the restriction of M to $S \setminus Z$.

Given matroid $M = (S, \mathcal{I})$ and $e \in S$, the <u>contraction</u> of M with respect to e is $M/e := (M^* \setminus e)^*$.

For $Z \subseteq S$, $M/Z = (M^* \setminus Z)^*$.

-or-

$M/e = (S-e, \mathcal{I}')$ where

$\mathcal{I}' = \{I \subseteq S-e \mid I+e \in \mathcal{I}\}$.

A matroid M' is a <u>minor</u> of M if M' is obtained from M by a sequence of contractions & deletions.

Problem: Given matroid $M=(S, \mathcal{I})$ & $w: S \to \mathbb{R}$.

Want to output $\underset{I \in \mathcal{I}}{\operatorname{argmax}}\, w(I)$.

Algorithm:

Discard all $e \in S$ where $w(e) \leq 0$ or e is a loop.

Let $= \{e_1, \ldots, e_n\}$ s.t. $w(e_1) \geq w(e_2)$
$\geq \ldots$
$\geq w(e_n)$.

$X \leftarrow \emptyset$

for $i \leftarrow 1$ to n
  if $X + e_i \in \mathcal{I}$
    $X \leftarrow X + e_i$

return X

—

Thm: Algorithm returns an optimal solution to max weight independent set of a matroid.

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Claim: There exists an optimal solution that contains e_1.

Proof: Let I^* be a max weight ind. set,

If $e_1 \in I^*$, we are done.

o.w.

$I^* + e_1$ is not independent, because $w(e_1) > 0$ & I^* has max weight.

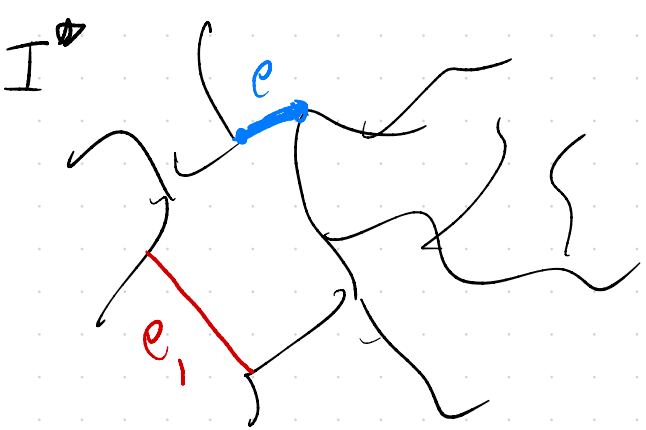
So $I^* + e_1$ has a circuit with some element e.

By a proposition on Monday,

$I^* - e + e_1 \in \mathcal{I}$.

$$w(I^* - e + e_1) = w(I^*) - w(e) + w(e_1)$$
$$\geq w(I^*)$$

So $I^* - e + e_1$ proves the claim.



So correct to add e_1. Then we want max ind. set in M/e_1 which we find by induction.

Remarks: If every element has positive weight, you find a base.
So you can add a large value to all element weights to guarantee alg returns a max weight base.

Can solve min weight base by negating weights & then finding a max weight base.
(we can solve min spanning tree - Krwska 1!)

Thm: The convex hull of the characteristic vectors of the ind. sets of $M = (S, \mathcal{I})$ with rank function $r_M : 2^S \to \mathbb{Z}_+$ is given by

$$x(A) \leq r_M(A) \quad \forall A \subseteq S,$$
$$x \geq 0.$$

Strong Base Exchange Theorem:

Let B, B' be two bases of $M = (S, \mathcal{I})$. For any $x \in B \setminus B'$, there exists a $y \in B' \setminus B$ s.t. both

$$B - x + y \text{ & } B' - y + x \text{ are bases.}$$

Applies to any pair of independent

sets of same size.

Given a set function $f: 2^S \to \mathbb{R}$

is <u>submodular</u> if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$$\forall\ A, B \subseteq S.$$

- or -

$$f(A + e) - f(A) \geq f(B + e) - f(B)$$

$$\forall\ A \subseteq B \subseteq S \text{ \& } e \in S \setminus B.$$

diminishing marginal returns

Examples:

Given an undirected graph $G = (V, E)$ with non-negative edge capacities $c: E \to \mathbb{R}_+$, the <u>cut function</u>

$f: 2^V \to \mathbb{R}_+$ with $f(A) = c(\delta(A))$

is submodular, non-negative, symmetric, but not monotone.

Given a hypergraph $G = (V, E)$

(edges may be arbitrary subsets of vertices), & $c: E \to \mathbb{R}_+$, the cut function $f(A) = c(\delta(A))$

where $\delta(A) := \{e \in E \mid e \cap A \neq \emptyset$ and $e \cap (S \setminus A) \neq \emptyset\}$,

is submodular, non-neg, symmetric, not monotone.

In a directed $D = (V, A)$ with $c: A \to \mathbb{R}_+$, $f(A) = c(\delta^+(A))$ is submodular, but not symmetric or

monotone.