A matroid M is a pair $(S, \mathcal{I})$ where S is a finite ground set and $\mathcal{I} \subseteq 2^S$ is a collection independent sets which satisfy the following properties:

1) $\mathcal{I}$ is non-empty; in particular $\emptyset \in \mathcal{I}$.

2) $\mathcal{I}$ is downward closed: $X \in \mathcal{I}$ & $Y \subseteq X$ implies $Y \in \mathcal{I}$.

3) If $X, Y \in \mathcal{I}$ & $|X| < |Y|$, then there is some $y \in Y \setminus X$ s.t.

$X + y \in \mathcal{I}$.

Examples: Vector Matroid (Linear Matroid)

Let M be an $m \times n$ matrix with entries in some field $\mathbb{F}$ & let v_i denote the ith column in M.

$S = \{1, 2, \ldots, n\}$

$\mathcal{I} = \{I : I \subseteq S, \{v_i\}_{i \in I}$ are linearly independent$\}$

Property 3: Suppose $X, Y \in \mathcal{I}$ & $|X| < |Y|$.

If no choice of $y \in Y \setminus X$ such that $X + y \in \mathcal{I}$, then $\{v_y\}_{y \in Y}$ is in span

of $\{v_x\}_{x \in X}$. So, $|Y| \leq |X|$, a contradiction.

Graphic Matroid (Cycle Matroid)

Let $G = (V, E)$ be an undirected multigraph.

$\mathcal{I} = \{I : I \subseteq E, I \text{ induces a forest}\}$.

$M = (E, \mathcal{I})$ is a matroid.

Prop 3: Let $X, Y \in \mathcal{I}$ with $|X| < |Y|$.

Let $V_1, V_2, \ldots, V_{k(X)}$ be vertex sets from components of $G[X]$.

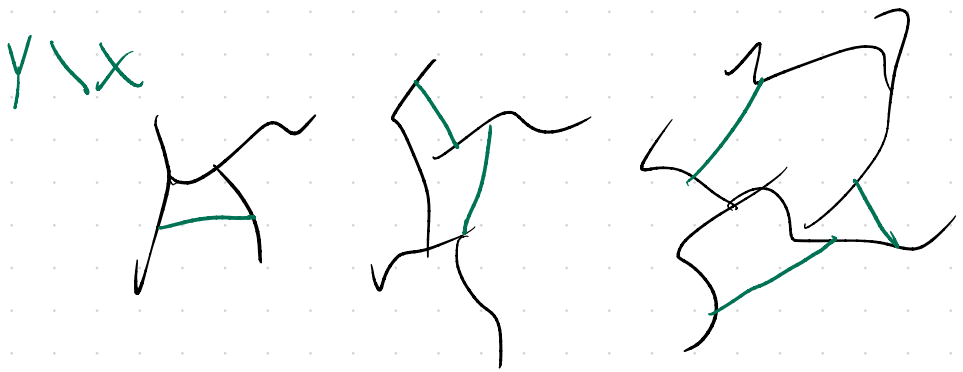
($k(X)$: # components in $G[X]$.)

Each component is a tree.

If $y \in Y \setminus X$ exists s.t. y connects two

of these components, $x + y \in I$.

But what if no such y?



X

So $k(Y) \geq k(X)$.

$\Rightarrow |X| = |V| - k(X) \geq |V| - k(Y) = |Y|$

a contradiction.

Uniform Matroid:

S: any finite non-empty set

$\mathcal{I} = \{I : I \subseteq S, |I| \leq k\}$ for some positive integer.

Partition Matroid

Let $S_1, S_2, \ldots, S_h$ be a partition of S & $k_1, k_2, \ldots, k_h$ be positive integers

$\mathcal{I} = \{I : I \subseteq S, |I \cap S_i| \leq k_i$ for all $1 \leq i \leq k\}$.

Laminar Matroids:

Let $\mathcal{F}$ be a laminar family on S ($X, Y \in \mathcal{F} \Rightarrow$ either $X \cap Y = \emptyset$ or $X \subseteq Y$ or $Y \subseteq X$)

s.t. for each $x \in S$ there is a $X \in \mathcal{F}$ s.t. $x \in X$.

Assign a positive integer $k(X)$ for each $X \in \mathcal{F}$.

$$\mathcal{I} = \{I : I \subseteq S, |I \cap X| \leq k(X) \quad \forall X \in \mathcal{F}\}.$$

Transversal Matroid:

Let $G = (V, E)$ be bipartite with sids V_1 & V_2.

Let $\mathcal{I} = \{I : I \subseteq V_1$, exists a matching in G that covers $I\}$.

$M = (V_1, \mathcal{I})$

Matching Matroid:

$G = (V, E)$ be any undirected graph.

$\mathcal{I} = \{I, I \subseteq V$, some matching covers $I\}$.

$M = (S, \mathcal{I})$ a matroid.

If $X \subseteq S$ but $X \notin \mathcal{I}$, then X is a dependent set.

A loop is any $x \in S$ s.t. $\{x\}$ is dependent.

A base is an inclusion wise maximal independent set.

Prop: If B & $\hat{B}$ are bases, then $|B| = |\hat{B}|$.

Lemma: Let B & $\hat{B}$ be bases of M. Let $x \in \hat{B} \setminus B$.

Then, exists $y \in B \setminus \hat{B}$ s.t.
$\hat{B} - x + y$ is a base of M.

Given $\hat{S} \subseteq S$, let $\hat{\mathcal{I}} = \{I : I \subseteq \hat{S}, I \in \mathcal{I}\}$.

$\hat{M} = (\hat{S}, \hat{\mathcal{I}})$ is a matroid called the <u>restriction</u> of M to $\hat{S}$.

Given $M = (S, \mathcal{I})$ & $\hat{S} \subseteq S$, $\hat{B}$ is a <u>base</u> for $\hat{S}$ if $\hat{B}$ is a base of $\hat{M}$, the restriction of M to $\hat{S}$.

Prop: Let $B \subseteq X$ be a base for $X \subseteq S$. For any $Y \supseteq X$, there is

a base $\hat{B}$ for Y that contains B.

Given $M = (S, \mathcal{I})$, a <u>circuit</u> is a minimal dependent set.

i.e. if C is a circuit, then

$\forall x \in C,\ C - x \in \mathcal{I}$.

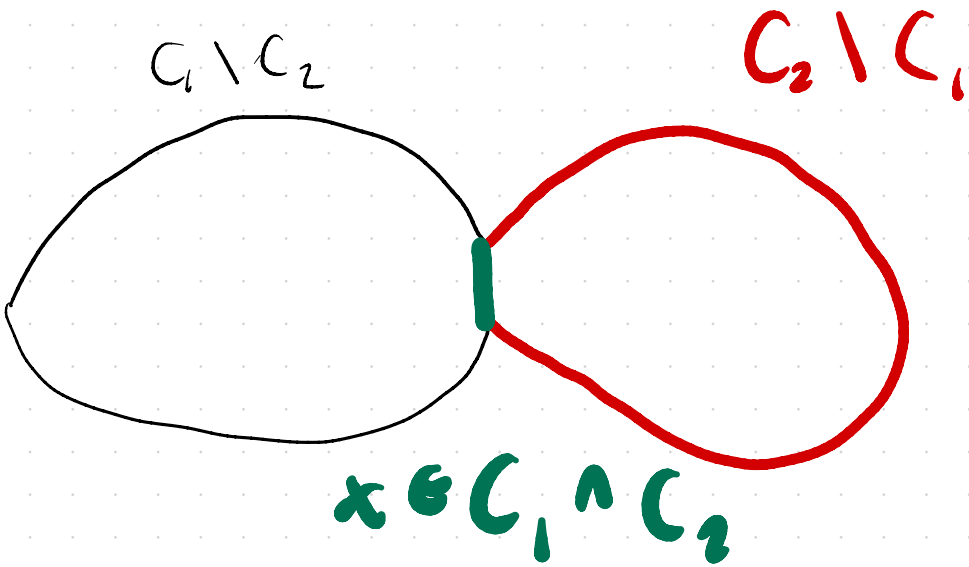
Lemma: Let $C_1 \neq C_2$ be two circuits s.t. $C_1 \neq C_2$ & $x \in C_1 \cap C_2$.

Then for every $x_1 \in C_1 \setminus C_2$ there is a circuit C s.t.

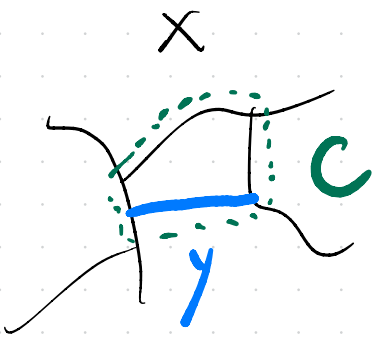
$x_1 \in C$ & $C \subseteq C_1 \cup C_2 - x$.

In particular, $C_1 \cup C_2 - x$ contains

a circuit.



Corollary: Let $M = (S, \mathcal{I})$ be a matroid. If $X \in \mathcal{I}$ & $y \notin X$, then either $X + y \in \mathcal{I}$ or there is a unique circuit C in $X + y$. Also, for each $\hat{y} \in C$, $X + y - \hat{y} \in \mathcal{I}$.



Corollary: Let B & $\hat{B}$ be bases. Let $\hat{x} \in \hat{B} \setminus B$, then $\exists x \in B \setminus \hat{B}$ such that $B + \hat{x} - x$ is a base.

Proof: From previous corollary. $x \in B \setminus \hat{B}$ is any of the $\hat{y} \in C$.

Let $M = (S, \mathcal{I})$ be a matroid.

The <u>rank function</u>, denoted r_M of M is $r_M : 2^S \to \mathbb{Z}_+$ where for $X \subseteq S$, $r_M(X)$ is the size of a maximal independent set in X.

Prop:

1) $0 \leq r_M(X) \leq |X|$ for all $X \subseteq S$.

2) r_M is <u>submodular</u>:
for any $X, Y \subseteq S$

$$r_m(X \cup Y) + r_m(X \cap Y) \leq r_m(X) + r_m(Y)$$