Given a graph $G=(V,E)$ with a subset of vertices $T \subseteq V$,

a subset $J \subseteq E$ is a T-join if T are the odd degree vertices of graph (V,J).

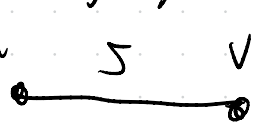
Thm: Given edge costs $c: E \to \mathbb{R}$, we can find a min-cost T-join in strongly polynomial time.

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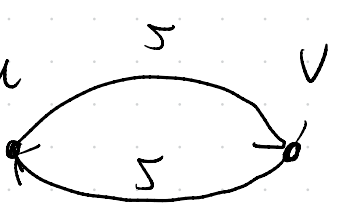
Shortest Paths.

Given $G=(V,E)$, $c: E \to \mathbb{R}$, $s,t \in V$, what is the shortest (simple) path from s to t?

Typically, you would



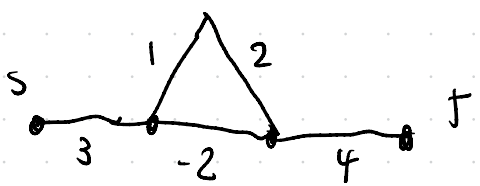
=>



But if edge has negative weight



Negative length cycle, so Bellman-Ford will fail!



Idea: Find a min-cost $\{s,t\}$-join.

Claim: Min-cost $\{s,t\}$-join is the shortest path if no negative length cycles.

"Proof": If T-join is not an $\{s,t\}$-path, it contains a cycle. Can remove cycle without increasing cost of T-join.

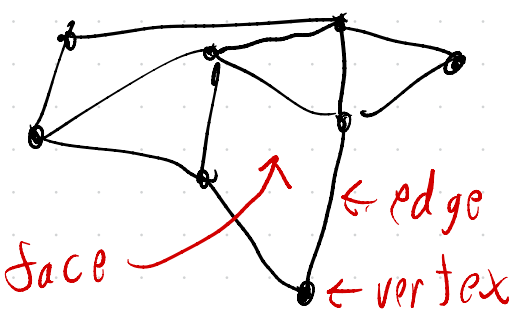


Max-cut in planar graphs

Max-cut: Given $G=(V,E)$ with $c: E \to \mathbb{R}^+$. Want a cut $S \subseteq V$ s.t. $c(\delta(S))$ is <u>maximized</u>.

NP-hard :

Planar graphs: Can map each vertex to a point in the plane. Edges as curves between endpoints. Edges disjoint except at their own vertices' endpoints.

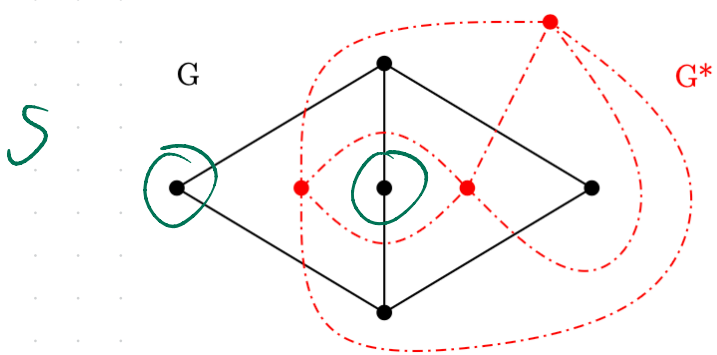


Face: maximal subset of plane - graph embedding

Dual graph: Each face of primal G becomes a vertex. $G^* = (V^*, E^*)$

Each edge between two faces becomes an edge between the dual vertices.

Can embed s.t. each primal vertex becomes a dual face.



Fact: For any $S \subseteq V$, the dual of $\delta(S)$ is a collection of edge-disjoint cycles in G^* & vice versa.

i.e. dual of max-cut (edges) is
a max weight $\emptyset$-join in the dual.

So strongly poly time in planar graphs.
(Algorithm by Hadlock)

Metric-TSP:

Traveling Salesperson Problem (TSP):

Given undirected $G=(V,E)$ with costs $c: E \to \mathbb{R}^+$.

Goal: Find a Hamiltonian cycle (simple cycle touching each vertex) with minimum weight.

NP-hard: Hard to find any Hamiltonian cycle. Much less cheapest one.

Metric-TSP:

Two variants: 1) G is complete & edge weights satisfy triangle inequality $c(uw) \leq c(uv) + c(vw)$ for any $u, v, w \in V$.

2) Find a cheapest tour touching each vertex at least once.

Still NP-hard.

So we'll try for an approximation algorithm.

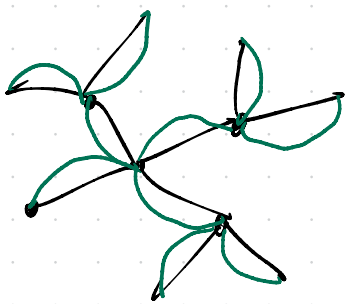
OPT: optimal tour cost

An α-approximation algorithm finds a tour of cost $\leq \alpha \cdot OPT$.

2-approximation algorithm:

(2nd variant)

1) Compute the minimum spanning tree.



2) Double up every edge in min spanning tree.

3) Return Eulerian tour in doubled tree. (Every vertex has even degree.)

Strongly poly time!

Claim $c(MST) \leq OPT$: Optimal tour is a connected spanning subgraph & MST is cheapest such thing.

We return a tour of cost $2c(MST) \leq 2OPT$. ✓

(Christofide '76): 3/2-approximation.

Compute MST H.

$T \leftarrow$ odd degree vertices of H.

$J \leftarrow$ min cost T-join

Return Eulerian tour in $H \cup J$

(some edges may be doubled)

Strongly poly time!

Lemma: Let T be any subset of vertices with $|T|$ even. There is a T-join of cost $\leq OPT/2$.

Proof: Consider first variant of Metric-TSP.
Optimal tour is Hamiltonian (no repeated vertices)

Let $\langle v_1, v_2, \ldots v_{2k-1}, v_{2k} \rangle$ be the subsequence of the optimal tour restricted to T.

$\{v_1v_2, v_3v_4, \ldots, v_{2k-1}v_{2k}\}$ is a T-join.

$\{v_2v_3, v_4v_5, \ldots, v_{2k}v_1\}$ is a T-join.

Total cost of all these edges is $\leq OPT$.
So one of the two sets has cost $\leq \frac{OPT}{2}$.

Tour we return has cost
$= c(H) + c(J) \leq OPT + OPT/2$
$= 3/2\,(OPT)$

KKG '21: $(3/2 - \varepsilon)$-approx in poly time for a small (but fixed!) $\varepsilon > 0$.