Route inspection (Guan's route problem)

(Chinese postman problem)

most common name

Given undirected $G=(V,E)$ with costs/weights $c: E \to \mathbb{R}^+$ (non-negative).

A Chinese postman tour starts at an arbitrary vertex & traverses every edge in E. We *may* use some edges more than once.

Want a tour of minimum total cost.

Easy case: If every vertex has even degree (it is Eulerian), we can take an Eulerian tour that visits each edge exactly once. at cost $\sum_{e \in E} c(e)$.

Let $T \subseteq V$ be the odd degree vertices.

Fact: $|T|$ is even (handshake lemma)

Consider optimal tour & let it visit each edge e $x(e) \geq 1$ times.

If we replace each e with $x(e)$ copies



New bigger graph is Eulerian.

Suppose some $x(e) > 2$...



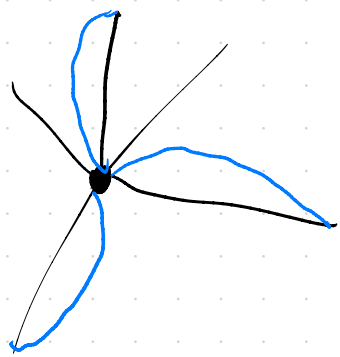
(multigraph)

We can reduce any $x(e) > 2$ by 2 to get a cheaper tour.

So, $x(e) \in \{1, 2\}$ in optimal solution.

So lets just find the edges we'd need to use twice.

Let E' be those edges we use twice.



If v has even degree in G, we need $|\delta(v) \cap E'|$ to be even so multigraph can be Eulerian.

If v had odd degree, we need $|\delta(v) \cap E'|$ to be odd.

$G=(V,E)$ &

Given any $T \subseteq V$, a <u>T-join</u> is a subset $J \subseteq E$ s.t.:

T is the set of odd degree vertices in graph (V, J).

Given $c: E \to \mathbb{R}_+$, also we want to find min cost T-joins.

(for route inspection T is odd degree vertices of input graph G)

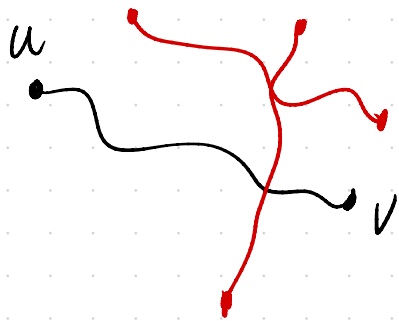
Proposition: The min cost T-join J for $c: E \to \mathbb{R}^+$ consists of the symmetric difference of $\frac{1}{2}|T|$ paths connecting disjoint pairs of vertices from T.

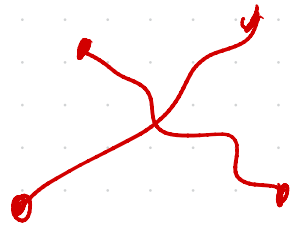
Proof: There is a path between some $u, v \in T$ by handshake lemma.

$J' := J \triangle P$ & $T' := T \triangle \{u, v\}$.

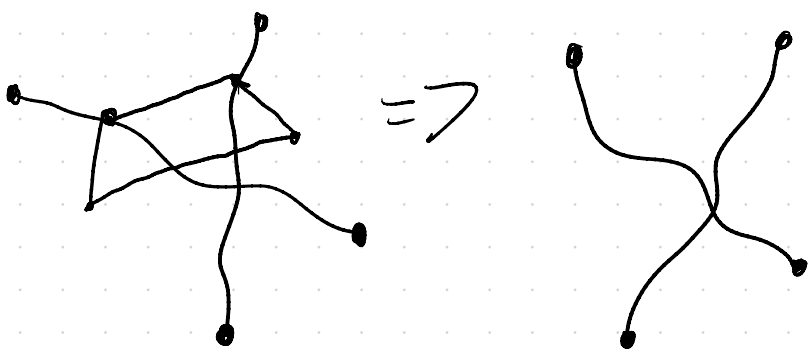
J is a T'-join.

Inductively, J' is a set of $\frac{1}{2}|T'|$ paths.





Also, no cycle in J: We could remove it & have a cheaper T-join.



Finally, those are shortest path.

If J has a not shortest (u,v)-path P

let P' be shortest & use $J \triangle P \triangle P'$ instead.



Algorithm for min cost T-join (with $c: E \to \mathbb{R}^+$)

For each pair $u,v \in T$ let $w(uv)$ be distance from u to v with edge lengths c.

Let P_{uv} be the shortest (u,v)-path.

Let H be the complete graph on T with edge weights $w(uv)$.

Compute a min weight perfect matching M in H.

Let $J = \triangle_{uv \in M} P_{uv}$. Output J.

Strongly poly time.

$\Rightarrow$ Route inspection in strongly poly

time.

Negative costs:

E^-: all negative cost edges

Ideally, E^- is a T-join (& therefore min cost!)

But what if it isn't?

Let T' be odd degree vertices in (V, E^-).

If $T' \neq T$, we want to "fix" E^-.

Let J'' be a $(T \triangle T')$-join.

Claim: $E^- \triangle J''$ is a T-join.

because $(E^- \triangle J'')$ is a $(T \triangle T') \triangle T' = T$-join

To find best J'': imagine picking edges from $E \setminus E^-$ at positive costs removing edges from E^-, also at postive cost.

Algorithm for min cost T-join with negative edge costs.

$E^- \leftarrow$ edges with negative costs

$T' \leftarrow$ set of odd degree vertices in (V, E^-).

$d: E \Rightarrow \mathbb{R}^+$ s.t. $d(e) = -c(e)$ if $e \in E^-$

$d(e) = c(e)$ if $e \notin E^-$

Compute a min cost $(T \triangle T')$ join J'' with non-negative cost d

Output $J = J'' \triangle E^-$.