Primal-Dual for min weight perfect matching given

$G = (V, E)$

weights: $w : E \to \mathbb{R}_+$

Primal

$$\min \sum_{e \in E} w(e) \cdot x(e)$$

$$x(\delta(v)) = 1 \qquad \forall v \in V$$

$$x(\delta(U)) \geq 1 \qquad \forall U \subset V, |U| \geq 3, |U| \text{ odd}$$

$$x(e) \geq 0 \qquad \forall e \in E$$

Dual $\max \sum_{\substack{U \subset V \\ |U| \text{ odd}}} \pi(U)$

$$\sum_{\substack{U \subset V \\ |U| \text{ odd} \\ e \in \delta(U)}} \pi(U) \leq w(e) \quad \forall e \in E$$

$$\pi(U) \geq 0 \quad \forall U \subset V, \ |U| \geq 3, \ |U| \text{ odd}$$

Algorithm:

$\Omega \leftarrow \{\{v\} \mid v \in V\}$,

$\pi(U) = 0 \quad \forall\, U,\ |U|$ odd

$M \leftarrow \emptyset$

$G' \leftarrow G_\pi$ ← G_π: induced graph of tight edges

while M is not a perfect matching in G'

$X \leftarrow M$-exposed edges of G'

$P \leftarrow$ an X-X, M-alternating walk in G'

if P is an M-augmenting path,

$M \leftarrow M \triangle E(P)$

continue

if P has an M-blossom B, (shrink)

$$U \leftarrow U_{v \in B} S_v$$

↑ vertices contracted to form v

$$\Omega \leftarrow \Omega \cup U$$

$$\pi(U) \leftarrow 0$$

$$G' \leftarrow G'/B$$

$$M \leftarrow M/B$$

continue

(no walk $P \Rightarrow M$ is maximum for G')

Compute $D(G')$, $A(G')$, $C(G')$ as in Edmonds-Gallai decomposition

($D(G')$: vertices avoided by some max matching)

(A(G') : neighbors of D(G') that aren't in D(G'). A Tutte-Berge witness set covered by every max matching of G')

(C(G') : the others)

$\epsilon \leftarrow$ largest value s.t.

$\pi(S_v) \leftarrow \pi(S_v) + \epsilon \quad \forall v \in D(G')$

$\pi(S_v) \leftarrow \pi(S_v) - \epsilon \quad \forall v \in A(G')$

remains dual feasible

if ϵ unbounded,

output NO PERFECT MATCHING

add newly tight edges to G_π

remove newly non-tight edges from G_π

for each $v \in A(G')$ with $|S_v| \geq 3$ and $\pi(S_v) = 0$,

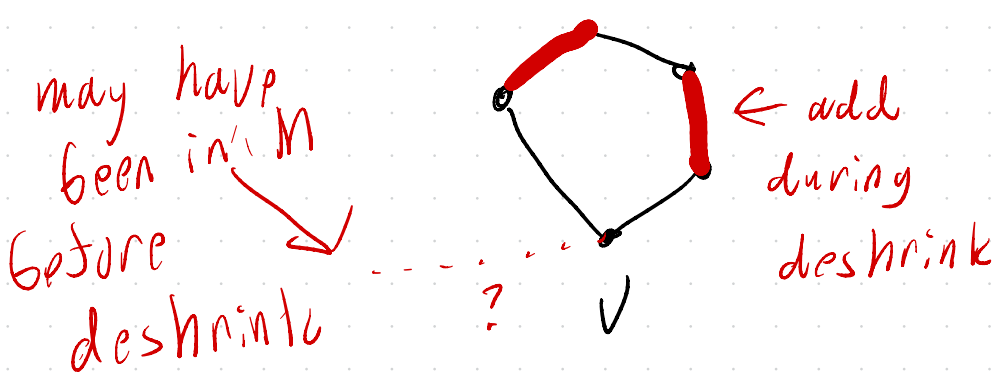
(deshrink)

$\Omega \leftarrow \Omega \setminus S_v$

unshrink the blossom that made v to update G'

B_{S_v} ← last blossom used to create v

extend M by a perfect matching in $B_{S_v} - \{v\}$



extend M in G' to a perfect matching in $G_\Uparrow$ & output M

Correctness:

Lemma: At all times

- π is dual feasible
- Ω is laminar
- for each $U \in \Omega$,

H_U ← graph obtained from $G_\Uparrow[U]$ by contracting each maximal proper subset $S \subset U$ with $S \in \Omega$.

(the subgraph of G' when

we add u to J_2

has a Hamiltonian cycle (a cycle containing all of H_u's vertices) B_u

(it was the blossom we contracted)

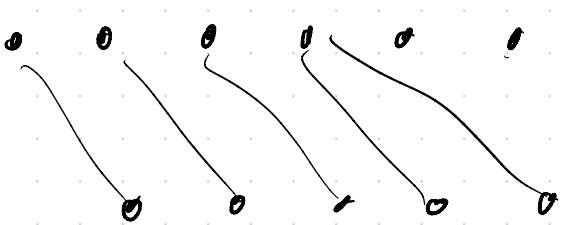
$\Rightarrow$ If M is a matching in G', there is a matching N in G_T with same # exposed nodes.

(we can repeatedly deshrink nodes & expand matching as we go)

$\Rightarrow$ If M is perfect in G', it can be extended to a perfect matching in G_ϕ

$\Rightarrow$ Algorithm returns a min cost perfect matching if it returns anything.

Prop: In any iteration, # M-exposed nodes does not increase & all edges of M remain tight.



If any ~~odd~~ component of $D(G')$ has ≥ 1 vertex, it would have had a blossom, so we don't go changing π this iter.

$\Rightarrow$ every edge of M connected to a member of $D(G')$ is also to a member of $A(G')$

the $+\epsilon$ and $-\epsilon$ balance out for those edges

other members of M go between vertices in $C(G')$

Lemma: If a set U is added to Ω (shrinking) we won't remove it until matching size of G has increased.

Proof: We shrink to a vertex v. There should be an X-v, M-alternating, even length path, because we can avoid v in a max matching.

Either v gets swallowed by a larger set U': we'll have to increase matching before U' gets deshrunk before

u is.

Or there is no matching augmenting
so still an X-v, M-alt
even path $\Rightarrow v \in D(G')$
$\Rightarrow v \notin A(G')$
so it can't be deshrunk

Claim: At most $|V|$ shrinkings &
deshrinkings between augmentation

Claim: If uv becomes tight
in a iteration & $u, v \in D(G')$ then
next iter shrinks or augments.

We just created a blossom opportunity

If instead $u \in D(G')$ & $v \in C(G')$ then we can find an M-alt path to v, meaning its not in C.

So... can only do last option $|V|$ times before a shrink or deshrink.

Can only shrink or deshrink $2|V|$ times

$\Rightarrow O(|V|)$ iters between

augmentations

$\frac{|V|}{2}$ augments

$\Rightarrow O(|V|^2)$ iters

$O(m)$ time per iter

Algo takes $O(n^2 m)$ time total.