Min Cost Perfect Matching in General Graphs (Primal-Dual).

Given $G = (V, E)$ + weights $w : E \to \mathbb{R}^+$.

By Edmonds:

Primal LP:

$$\min \sum_{e \in E} w(e) \cdot x(e) \quad \text{s.t.}$$

$$x(\delta(v)) = 1 \qquad \forall v \in V$$

$$x(\delta(U)) \geq 1 \qquad \forall U \subset V, |U| \geq 3, |U| \text{ odd}$$

$$x(e) \geq 0 \qquad \forall e \in E$$

Dual LP:

$$\max \sum_{U \subset V, |U| \text{ odd}} \pi(U) \quad \text{s.t.}$$

$$\sum_{U \subset V, |U| \text{ odd}, e \in \delta(U)} \pi(U) \leq w(e) \qquad \forall e \in E$$

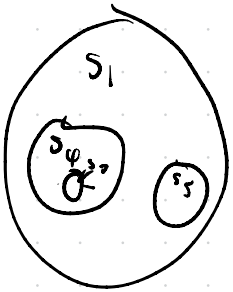
$\pi(U) \geq 0 \qquad \forall U \subset V,$
$|U|$ odd,
$|U| \geq 3$

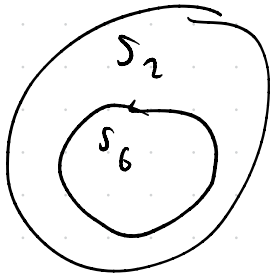
We'll maintain a feasible solution π to dual LP.

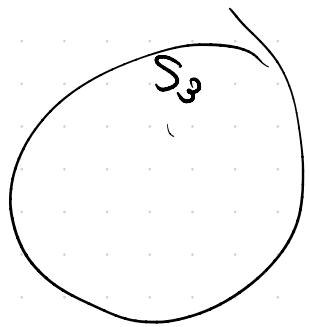
Repeat: Find a max matching using only edges e for which $\sum_{\substack{U \subset V, \\ |U| \text{ odd}, \\ e \in \delta(U)}} \pi(U) = w(e)$.

If perfect, return the matching.
o.w., improve dual solution & continue...

A family of sets $\{S_1, S_2, \ldots, S_p\}$ is laminar if no two sets properly cross,
i.e. for any two S_i & S_j, either
$S_i \cap S_j = \emptyset$, $S_i \subset S_j$, or $S_j \subset S_i$.







They form a rooted forest where S_i is a child of S_j if S_i is a maximal member of family with $S_i \subset S_j$.

We'll maintain a laminar family of odd subsets of V denoted by Ω.

Ω will always hold singletons (vertices).

$\pi(U) = 0$ for all $U \notin \Omega$.

Note: $|\Omega| \leq 2|V|$.

Say edge e is π-tight (or tight) if

$$\sum_{U \in \Omega,\ e \in \delta(U)} \pi(U) = w(e).$$

E_π : π-tight edges of E

G_π : graph induced by E_π

We'll maintain a graph G' where we contract each <u>maximal</u> member of Ω into a single vertex.

For each $v \in G'$, let $S_v \in \Omega$ be the vertices contracted to make v.

Thm (Edmonds-Gallai): Given $G = (V, E)$, let

$D(G) := \{v \in V \mid$ some max matching misses $v\}$

$A(G) := \{v \in V \mid v$ is a neighbor of $D(G)$ but $v \notin D(G)\}$

$C(G) := V \setminus (D(G) \cup A(G))$.

Then.

1) Set $A(G)$ is a Tutte-Berge witness set.

2) $C(G)$ is even components of $G - A(G)$

3) $D(G)$ is odd components of $G - A(G)$

4) Each component in $G - A(G)$ is factor-critical.

Also: If any component of $D(G)$ has > 1 vertex, it has a blossom.

Algorithm:

Initialize: $\Omega \leftarrow \{\{v\} \mid v \in V\}$,

$\pi(u) \leftarrow 0 \ \forall \ u$,

$M \leftarrow \emptyset$ // a matching in G'

$G' \leftarrow G_\pi$.

while M is not a perfect matching in G'

$X \leftarrow M$-exposed vertices in G'

Find X-X, M-alternating walk P in G'

If P is an M-augmenting path

$M \leftarrow M \triangle E(P)$

continue while loop

If P has an M-blossom B

do shrinking by ...

$U \leftarrow \bigcup_{v \in B} S_v$ ← all vertices of G shrunk to form B

$\Omega \leftarrow \Omega \cup U$

$\pi(U) \leftarrow 0$

$G' \leftarrow G'/B$

$M \leftarrow M/B$

continue the loop

(No alternating walk $P \Rightarrow M$ is a maximum matching in G')

Compute $D(G')$, $A(G')$, & $C(G')$ as in Edmonds-Gallai decomposition

$\epsilon \leftarrow$ largest value s.t. $\pi(S_v) \leftarrow \pi(S_v) + \epsilon$ $\forall v \in D(G')$ &

$\pi(S_v) \leftarrow \pi(S_v) - \epsilon$

$\forall_v \in A(G')$

maintains dual feasibility of π in G.

(either $\pi(U)$ goes to 0 for some odd U $|U| \geq 3$

or some edge goes tight)

If ϵ is unbounded, output G has no perfect matching (dual has no finite optimal solution)

Add newly tight edges to G_π.

Remove no longer tight edges from G_π.

For each $v \in A(G')$ with $|S_v| \geq 3$ and $\pi(S_v) = 0$

Deshrink v by

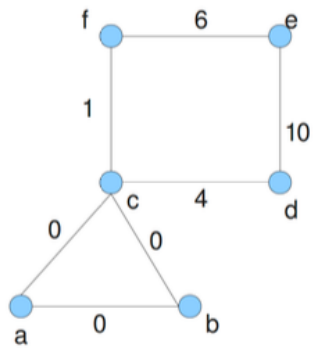
Remove S_v from Ω

Update G'

Extend M by a perfect matching in S_v's blossom $- \{v\}$.

Extend M to a perfect matching in G_π & return it.

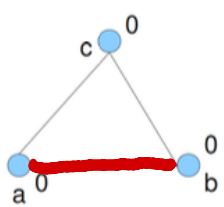
Example:



Initially, only 0 edges in G_π.

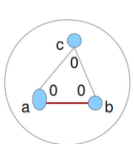


$\pi(d) = 0$



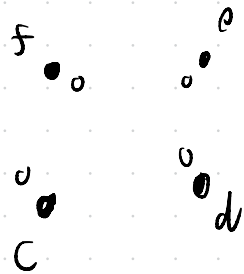
After iteration 1 (augment M).







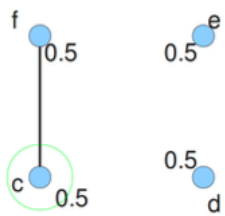
About to shrink blossom in iter 2.



End of iter 2 (classum shrunk to c)

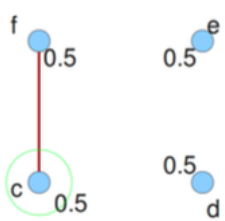
Everything in $D(G')$

Find $\varepsilon = 1/2$



End of iter 3 (we increased all $\pi(S_v)$ by $1/2$)

f_c went tight

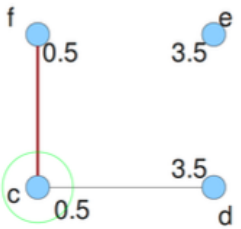


End of iter 4 (added f_c to M)

EG decomp has $D(G') = \{e, d\}$

$A(G') = \emptyset$

$C(G') = \{f, c\}$



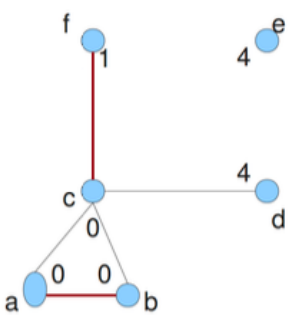
After iter 5 (ϵ was 3)

cd went tight

EG decomp has $D = \{f, e, d\}$

$A = \{c\}$

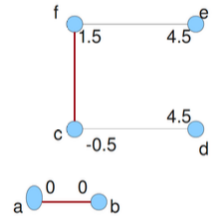
$C = \emptyset$



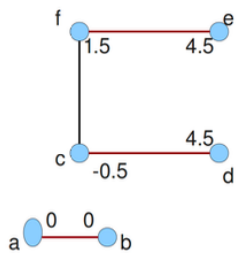
After iter 6, ϵ was 0.5

~~crash~~ Deshrunk

c.



After iter 7. ϵ was 0.5
fe went tight



After iter 8. We increased M
using path efcd.

Return M!