Primal LP $\min c^T x$

$Ax = b,$

$x \geq 0$

Recall, x & y satisfy complimentary slackness if

$x_i > 0$ iff $y^T a_i = c_i$ for all i

Dual LP $\max y^T b$

$y^T A \geq c$

Say constraint i is <u>tight</u> if $y^T a_i = c_i$.

Suppose we have a dual solution y_0.

Check if there is a solution to

$x' \geq 0, A' x' = b$ where x' contains the variables x_i s.t. $y_0^T a_i = c_i$.

Fewer variable than primal LP. no objective to optimize!

If x' exists, set x to x' padded with 0s. x & y satisfies complimentary slackness & are optimal!

But if no x'...

By Farkas' lemma, there is a z s.t.

$$z^T b < 0 \text{ & } z^T A \geq 0.$$

Let $y_1 = y_0 - \alpha z$ where $\alpha > 0$ is large as possible s.t. $y_1^T A \leq c$.

$y_1^T b > y_0^T b$, so we made progress.

(If no best α, then dual LP has no finite optimal solution & primal LP has no solutions at all.)

Augmenting paths max flow is an example (min cut is primal LP. max flow is dual)

Minimum Weight/Cost Perfect Matching:

Given a graph $G = (V, E)$ with weights $w: E \to \mathbb{R} \cup \{\infty\}$, find a perfect matching with total weight.

Can assume weights are non-negative.

Can reduce max weight matching to min weight perfect matching in poly time. (Homework?)

Bipartite graphs: $G = (A \sqcup B, E)$

ILP: $\min \sum_{(a,b) \in E} w(a,b)\, x(a,b)$ s.t.

Primal LP:

$$\sum_b x(a,b) = 1 \qquad \forall a \in A$$

$$\sum_a x(a,b) = 1 \qquad \forall b \in B$$

~~$x(a,b) \in \{0,1\}$ $\forall (a,b) \in E$~~

$x(a,b) \geq 0$ $\forall (a,b) \in E$

But TUM so...

Find a vertex solution to ↑

Dual LP $\max \sum_{a \in A} y(a) + \sum_{b \in B} y(b)$

$y(a) + y(b) \leq w(a,b) \quad \forall (a,b) \in E$

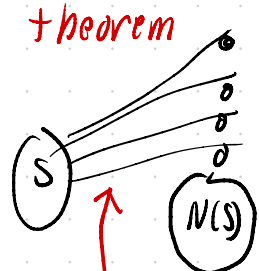
Given y, an edge (a,b) is tight if $y(a) + y(b) = w(a,b)$.

Algorithm 3 MinWeightPerfectMatching($G = (V, E), w$)

1: $y \leftarrow 0$
2: $E' \leftarrow$ set of tight edges
3: $M \leftarrow$ max cardinality matching for graph $G' = (V, E')$
4: **while** M is not a perfect matching **do**
5: let $G' = (V, E')$
6: let $S \subseteq A$ be such that $|S| > |N(S)|$
7: let $\epsilon = \min_{a \in S, b \in B \setminus N(S)} \{w(a,b) - y(a) - y(b)\}$
8: $\forall a \in S \; y(a) = y(a) + \epsilon$
9: $\forall b \in N(S) \; y(b) = y(b) - \epsilon$
10: update E', M
11: **end while**
12: **return** M

will exist by Hall's theorem

neighborhood of S in $G' = (V, E')$



non-tight edges of G

Edges (a,b) s.t. $a \in S$, $b \in B \setminus N(S)$ are not tight, so $w(a,b) - y(a) - y(b) > 0$

$N(S)$ remains unchanged. the edges E' stay tight

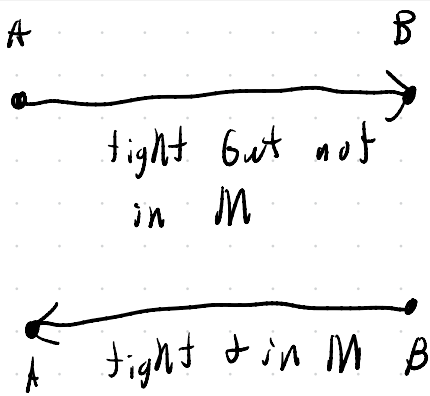
the edge defining ϵ becomes tight. Value of solution increases by $\epsilon(|S| - |N(S)|)$.

+ stays feasible.

Pseudo-poly time: If costs are integers,
then $\epsilon \geq 1$ each iter:
$\leq$ optimal value iterations

Algorithm 4 MinWeightPerfectMatchingPD($G = (V, E), w$)

$\forall b \in B\ y(b) \leftarrow 0$
$\forall a \in A\ y(a) \leftarrow \min_b \{w(a, b)\}$
$E' \leftarrow$ set of tight edges
$M \leftarrow$ max cardinality matching for graph $G' = (V, E')$
while M is not a perfect matching **do**
    let $E_{dir} \leftarrow \{e$ directed from A to $B \mid e \in E', e \notin M\} \bigcup$
                $\{e$ directed from B to $A \mid e \in E', e \in M\}$
    let $D = (V, E_{dir})$ ▷ D is a directed graph
    let $L \leftarrow \{v \mid v$ is reachable in D from an unmatched vertex in $A\}$
    let $\epsilon = \min_{a \in A \cap L, b \in B \setminus L} \{w(a, b) - y(a) - y(b)\}$
    $\forall a \in A \cap L\ y(a) = y(a) + \epsilon$
    $\forall b \in B \cap L\ y(b) = y(b) - \epsilon$
    update E', M
end while
return M



(paths in D from exposed A to exposed B are augmenting. so no such path, because M is maximum)

Claim: $C = (A \setminus L) \cup (B \cap L)$ is a vertex cover in $G' = (V, E')$ of size $|M|$.

Proof: If not a cover, some $(a, b) \in E'$ is not

covered. $a \in L \cap A$ & $b \in B \setminus L$.

If $a \to b$ in D $((a,b) \notin M)$
$a \in L$ but b is not? $\perp$

If $b \to a$ in D $((a,b) \in M)$
$a \in L$, but $a \to b$ on path to a, so
$b \in L$ $\perp$.

- All of $A \setminus L$ is matched.
- All of $B \cap L$ is matched because no augmenting path.
- No $(a,b) \in M$ s.t. $a \in A \setminus L$ & $b \in B \cap L$ (see above).

So every vertex in C is matched, & no edge is matching two members of C, so, $|C| \leq |M|$.

But $|C| \geq |M|$, so $|C| = |M|$.

Claim implies no tight $(A \cap L) - (B \setminus L)$ edge, so, $\epsilon > 0$.
(& can directly argue no y has higher value)

Claim: Algorithm terminates in $O(|V|^2)$ iterations.

Can do each iteration in $O(|V|^2)$ time, so $O(|V|^4)$ time total

($O(|V|^3)$ if more careful)