Tutte-Berge: Given graph G, the size of the max cardinality matching, denoted by $r(G)$, is

$$r(G) = \min_{U \subseteq V} \frac{1}{2}\left(|V| + |U| - o(G-U)\right)$$

$o(G-U)$: # odd components

where $o(G-U)$ is # components in $G[V \setminus U]$ with odd cardinality.

$G[V \setminus U]$: subgraph induced by $V \setminus U$.

The "worst" set U: Tutte-Berge witness set.

- All vertices in U covered by every max matching in G.



- if K is the vertex set of any component of $G-U$,

every max matching covers at least $2 \cdot \lceil |C|/2 \rceil$ vertices of k.

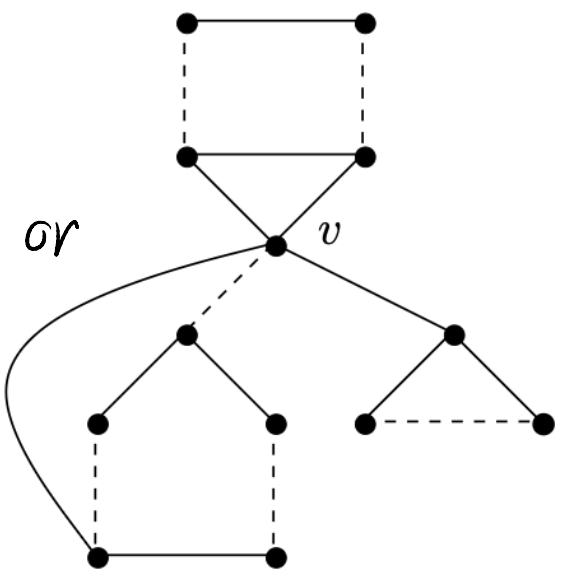
There could be multiple witness sets.

Can use $U = \emptyset$ or

$U = \{v\}$.

$(r(G) = 6)$

more info



13 nodes

Want a "best" choice for U.

Graph $G = (V, E)$ is factor-critical if G has no perfect matching, but for each $v \in V$, $G - v$ has a perfect matching.

$\Rightarrow G$ has odd # vertices

Thm: G is factor-critical iff for every v there is a max matching that misses v.

Proven on second lecture!

Proposition: Let C be an odd cycle in G. If G/C is factor-critical, then G is as well.

↑ shrink C to a vertex

Thm (Edmonds-Gallai Decomposition)

Given a graph $G = (V, E)$ let

- $D(G) := \{v \in V \mid$ there is a max matching that misses $v\}$
- $A(G) := \{v \in V \mid v$ is a neighbor of $D(G)$ but $v \notin D(G)\}$

- $C(G) := V \setminus (D(G) \cup A(G))$

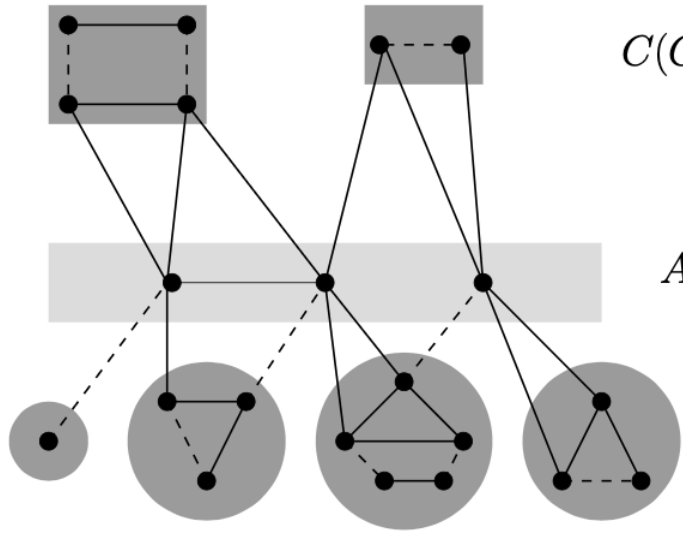
Then,

1) $U = A(G)$ is a Tutte-Berge witness set for G.

2) $C(G)$ is the union of even components of $G - A(G)$

3) $D(G)$ is the union of odd components of $G - A(G)$

4) Each odd component of $G - A(G)$ is factor critical.



---: in a max matching

Corollary: A graph is factor-critical iff $U = \emptyset$ is the unique Tutte-Berge witness set for G.

Let M be any max matching in G & let X be the M-exposed nodes.

↖ not covered by M

$Even(G,M) := \{v \in V \mid$ there is an even length (# edges) M-alternating (X,v)-path$\}$

$Odd(G,M) := \{v \in V \mid$ there is an M-alternating (X,v)-path$\} \setminus Even(G,M)$

$Free(G,M) := \{v \in V \mid$ There is no M-alternating (X,v)-path$\}$

Lemma: For any max matching M in G

1) $Even(G,M) = D(G)$
2) $Odd(G,M) = A(G)$
3) $Free(G,M) = C(G)$

Proof:

1) Suppose $v \in Even(G,M)$, Then there is an even length M-alternating (X,v)-path.



v is exposed in $M \Delta E(P)$, so $v \in D(G)$.

Suppose $v \in D(G)$. There is a max matching M_v missing v.

$M \triangle M_v$ contains an even length M-alternating (M,v)-path

$\Rightarrow v \in Even(G,M)$.

2) If $v \in Odd(G,M)$, let P be an M-alternating (x,v)-path.

P has odd length.

Let uv be last edge of path.

There is an even length M-alt path from x to u.

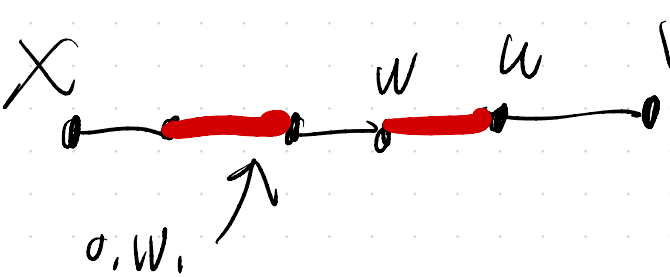


$\Rightarrow u \in D(G)$.

$\Rightarrow v \in A(G)$.

Suppose $v \in A(G)$. Let $uv \in E$ s.t. $u \in D(G) = Even(G,M)$.

So there is an even length M-alt (X, a)-path P which ends in some $wu \in M$.



If $v \in P$, then $v \in Odd(G,M)$.

$P + uv$ is an M-alt (X, v) path of odd length, because $uv \notin M$, because $wu \in M$.

$= \exists v \in Odd(G, M)$.

3) $C(G) = V \setminus (D(G) \cup A(G))$

$= V \setminus (Even(G,M) \cup Odd(G,M))$

$= Free(G,M)$

Below: M is always a max matching.

Lemma: Each vertex in $A(G) \cup C(G)$ is covered by M & every $v \in A(G)$ is matched to some vertex in $D(G)$.

Corollary: Each component of $G[C(G)]$ is even and $|M \cap C(G)| = |C(G)|/2$.

Lemma: Each component H of $G[D(G)]$ satisfy:

1) Either $|V(H) \cap X| = 1$ & $|M \cap \delta_G(V(H))| = 0$, or $|M \cap \delta_G(V(H))| = 1$

2) H is factor-critical.

Proof of main theorem:

Each component of $G[D(G)]$ is factor-critical and therefore odd.

So... $G[C(G)]$ has even components of $G - A(G)$, & $G[D(G)]$ has odd components of $G - A(G)$.

Only thing left is $A(G)$ is a Tutte-Berge witness.

Need to show $|M| = \frac{1}{2}(|V| - |A(G)| - o(G - A(G)))$.

$|M| = \frac{1}{2}(|V| - |X|)$

Need to show $|X| + |A(G)| = o(G - A(G))$

$o(G - A(G)) - |X| = A(G)$ from previous lemmas, which implies the other equations