Let $G=(V,E)$ be a graph.

For $E' \subseteq E$, let $\chi^{E'}$ denote the characteristic vector of E' in $\mathbb{R}^{|E|}$.

$$\chi^{E'} \in \{0,1\}^{|E|}$$

ith G it is 1 if ith edge in E'

$$P_{\text{perfect_matching}}(G) := \text{convex hull}(\{\chi^M \mid M \text{ is a perfect matching in } G\})$$

$$P_{\text{matching}}(G) := \text{convex hull}(\{\chi^M \mid M \text{ is a matching in } G\})$$

If G bipartite

$P_{pm}(G)$ given by

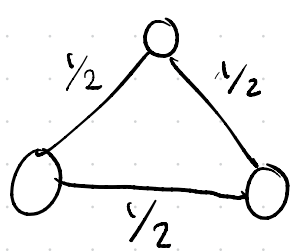
empty if no perfect matching

$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(e) \geq 0 \quad \forall e \in E$$

$P_m(G)$

$$x(\delta(v)) = 1 \quad \forall v \in V$$
$$x(e) \quad \forall e \in E$$

But if G is not bipartite,



Thm (Edmonds):

$P_{pm}(G)$ is determined by

$$x(e) \geq 0 \qquad e \in E$$

$$x(\delta(v)) = 1 \qquad v \in V$$

$$x(\delta(U)) \geq 1 \qquad U \subseteq V,\ |U| \geq 3 \text{ and } |U| \text{ is odd}$$

$$\sum_{e \in \delta(U)} x(e)$$

Proof:

Let $Q(G)$ be described by the inequalities.

$$P_{pm}(G) \subseteq Q(G)$$

Need to argue $Q(G) \subseteq P_{pm}(G)$

Suppose there is a graph G s.t. $Q(G) \not\subseteq P_{pm}(G)$. Choose G to minimize $|V| + |E|$.

By assumption, there is a vertex $x \in Q(G)$ s.t. $x \notin P_{pm}(G)$.

Claim: $x \in (0,1)^{|E|}$.

If $x(e) = 0$ for some $e \in E$, just remove e from G to get a smaller counterexample

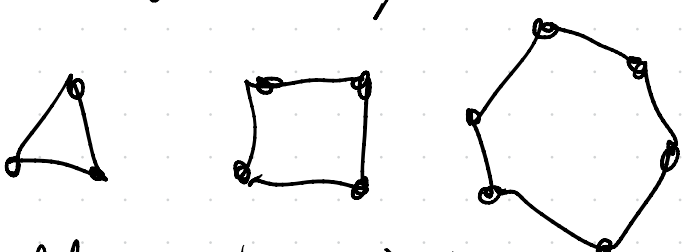
If $x(e) = 1$ for some $e \in E$, remove e and its endpoints for a smaller counterexample.

$|V|$ must be even, otherwise $Q(G) = \emptyset$ & x does not exist.

$0 < x(e) < 1$ for all e & $x(\delta(v)) = 1$ for all v, so $|\delta(v)| \geq 2$.

Suppose $|E| = |V|$, then $|\delta(v)| = 2$ for all vertices

$\Rightarrow$ G is a collection of vertex disjoint cycles.



No odd cycles. Otherwise $Q(G)$ is empty.

So G is even cycles $\Rightarrow Q(G) \subseteq P_{pm}(G)$

after all.

So $|E| \geq |V|$.

x is a vertex of $Q(G)$, so

$|E|$ tight inequalities (met with equality) determine x

$\Rightarrow$ there is at least one odd set $U \subseteq V$

s.t. $x(\delta(U)) = 1$ & $|U| \geq 3$.

These $|E|$ equalities are non-redundant

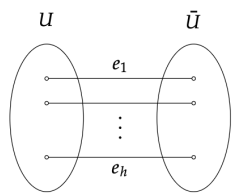
$\Rightarrow |V - U| > 1$

o.w. $V - U = \{v\}$ for some v

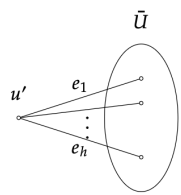
making $x(\delta(U)) = 1$ redundant with $x(\delta(v)) = 1$

Let $G' = G/\bar{U}$ be G with $\bar{U}$ shrunk to a single vertex u'.

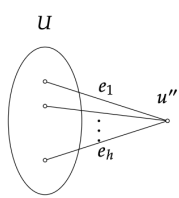
Let $G'' = G/U$ be G with $(V-U)$ shrunk to a vertex u''.



G



G'



G''

x' : x restricted to G'

x'' : x restricted to G''

$Q(G') = P_{pm}(G')$ + $Q(G'') = P_{pm}(G'')$

Because those graphs are

smaller,

x' is the convex combo of $P_{pm}(G')$'s vertices

similar for x''

x is rational $\Rightarrow$ x' & x'' are rational

$\Rightarrow$ $x' = \frac{1}{k} \sum_{i=1}^{k} \chi^{M'_i}$ where

same (large) k

each M'_i is a perfect matching of G'

$x'' = \frac{1}{k} \sum_{i=1}^{k} \chi^{M''_i}$

Let $e_1, e_2, \ldots, e_h$ be in $\delta(U)$.

$x'(\delta(u')) = 1$ & u' in every perfect matching of G', so

e_j is in exactly $k \cdot x'(e_j) = k \cdot x(e_j)$ of the matching $M'_1 \dots M'_k$.

Also e_j is in exactly $k \cdot x''(e_j) = k \cdot x(e_j)$ matching $M''_1 \dots M''_k$

Can renumber $M'_1 \dots M'_k$ & $M''_1 \dots M''_k$ so each M'_i & M''_i share exactly one edge from $e_1, \dots, e_n$.

$\Rightarrow M_i = M'_i \cup M''_i$ is a perfect matching in G.

But x wasn't supposed to be a perfect matching in G! $\perp$

Thm: $P_{matching}(G)$ is determined by

$$x(e) \geq 0 \qquad e \in E$$

$$x(\delta(v)) \leq 1 \qquad v \in V$$

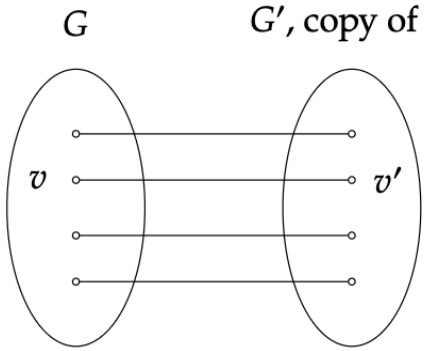
$$x(E[U]) \leq \frac{|U|-1}{2} \qquad U \subseteq V,\ |U| \text{ odd}$$

edges with both endpoints in U

Can reduce weighted matchings to weighted perfect matching:

Given $G=(V,E)$, make a copy $G'=(V',E')$.

Let $\tilde{G}(\tilde{V},\tilde{E})$ be with $\tilde{V} = V \sqcup V'$, $\tilde{E} = E \cup E' \cup \{vv' \mid v \in V\}$



Claim: Suppose M is a matching in G. There is a perfect matching $\tilde{M}$ in $\tilde{G}$ s.t. $\tilde{M} \cap E = M$.

Likewise, if $\tilde{M}$ is perfect in $\tilde{G}$, then $\tilde{M} \cap E$ is a matching in G.

Corollary: Max weight matching & max weight perfect matching are poly time equivalent.

Proof: Remove neg. edges from G. Make $\tilde{G}$ with $w(vv') = 0 \quad \forall v \in V$.

Find max weight perfect matching $\tilde{M}$ in $\tilde{G}$ & return $E \cap \tilde{M}$.

Weakly poly max weight perfect matching:

Would like to build $Q(G)$ & use an LP solver, but $Q(G)$ is exponentially large.

Use Ellipsoid method.

Need a separation oracle: given $x \in \mathbb{R}^{|E|}$, returns which is any inequality is violated.

$$x(e) \geq 0 \qquad e \in E$$
$$x(\delta(v)) = 1 \qquad v \in V$$
$$x(\delta(W)) \geq 1 \qquad |W| \text{ odd}, \ W \subseteq V$$

To know if there's a bad set U for a given x, need to solve minimum odd-cut.

Find the odd $U \subseteq V$ minimizing $x(\delta(U))$.

Compute Gomory-Hu tree $T = (V, E_T)$ with capacities given by x.

Among all odd cuts induced by edges of T, output one with min capacity.

Suppose $\delta(U^*)$ be the min odd cut. $\delta_T(U^*)$ is a subset of E_T.

Use best of those edges.

