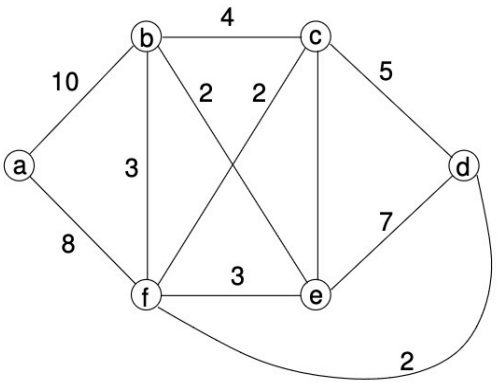
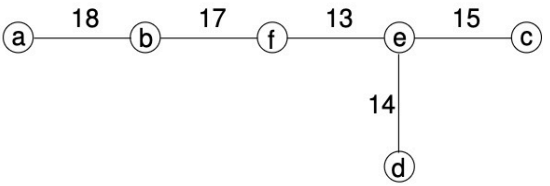


G



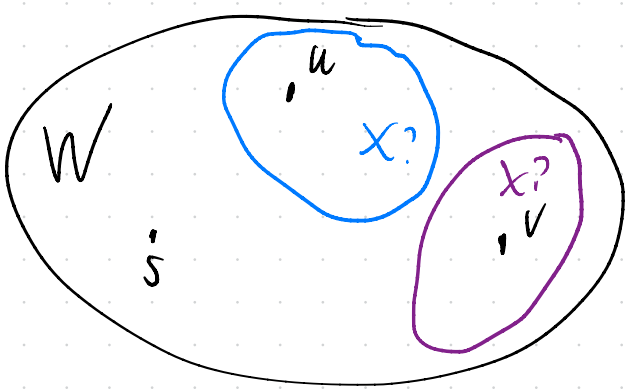
T

$\alpha_G(b,d) = \alpha_T(b,d) = 13$

Gomory-Hu trees: how to construct

Lemma: Let G be an edge-weighted graph & let $\delta(W)$ be a min s,t cut. For any $u, v \in W$, there is a min (u,v)-cut $\delta(X)$ s.t. $X \subseteq W$ (

$\cdot t$

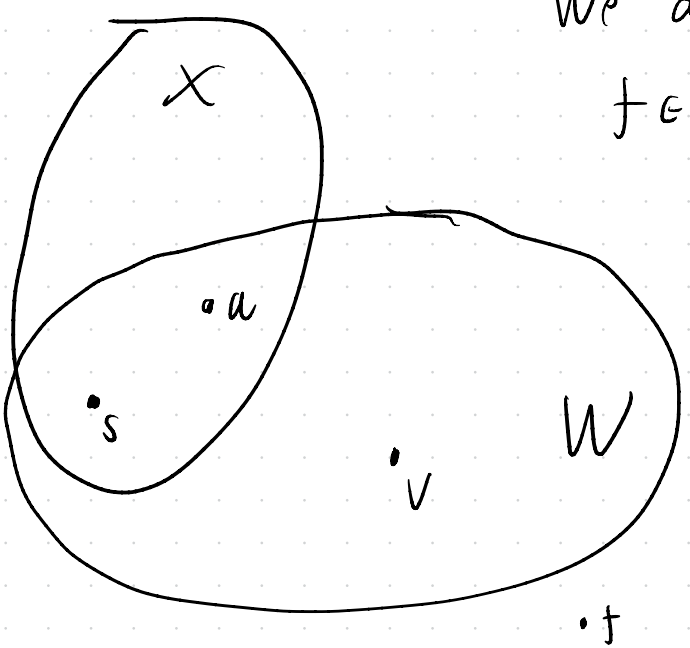


Proof: Let $\delta(X)$ be a (u,v)-cut.

Suppose it crosses W ($X \cap W \neq \emptyset$, $X \cap W \neq X$).

Without loss of generality,

$s \in W$, $s \in X$, $u \in X$.



We don't know if $t \in X$ yet.

Suppose $t \notin X$

$$c(\delta(X)) + c(\delta(W)) \geq c(\delta(X \cap W)) + c(\delta(X \cup W))$$

By submodularity

$\delta(X \cap W)$ is a (u,v)-cut
$\Rightarrow c(\delta(X \cap W)) \geq c(\delta(X))$

$\delta(X \cup W)$ is an (s,t)-cut
$\Rightarrow c(\delta(X \cup W)) \geq c(\delta(W))$

So... $c(\delta(X)) + c(\delta(W)) = c(\delta(X \cap W)) + c(\delta(X \cup W))$

$\Rightarrow \delta(X \cap W)$ is a min (u,v)-cut

Case 2: $t \in X$

By posi-modularity,

$$c(\delta(X)) + c(\delta(W)) \geq c(\delta(W \setminus X)) + c(\delta(X \setminus W))$$

$$c(\delta(W \setminus X)) \geq c(\delta(X))$$

$$c(\delta(X \setminus W)) \geq c(\delta(W))$$

$\Rightarrow$ $\delta(W \setminus X)$ is a min (u,v)-cut.

□

uncrossing

Let $G=(V,E)$, $R \subseteq V$.

A Gomory-Hu tree for R in G

is a pair with

- a tree $T=(R, E_T)$
- a partition $(C_r \mid r \in R)$ of V

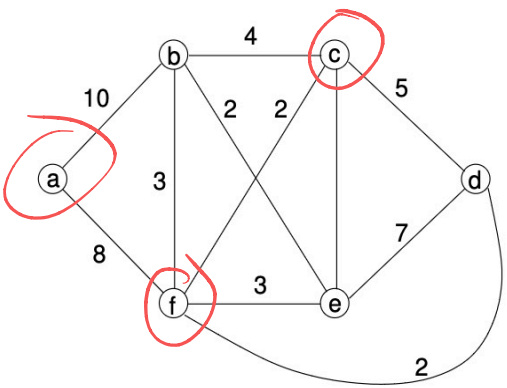
s.t.

1) For all $r \in R$, $r \in C_r$

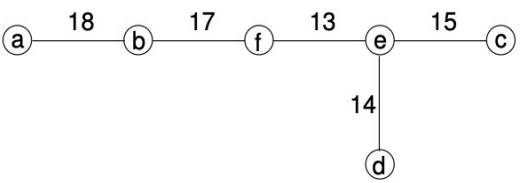
2) For all $st \in E_T$, $T-st$ induces a $\min^{(s,t)}$ cut in G defined by

$$\delta(U) = \bigcup_{r \in X} C_r$$

where X is the vertex set for one component of $T - st$.

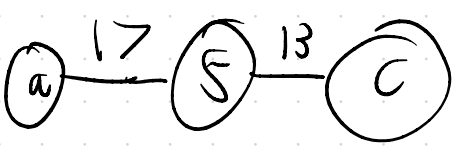


G



T for G

$R = \{c, d, f\}$



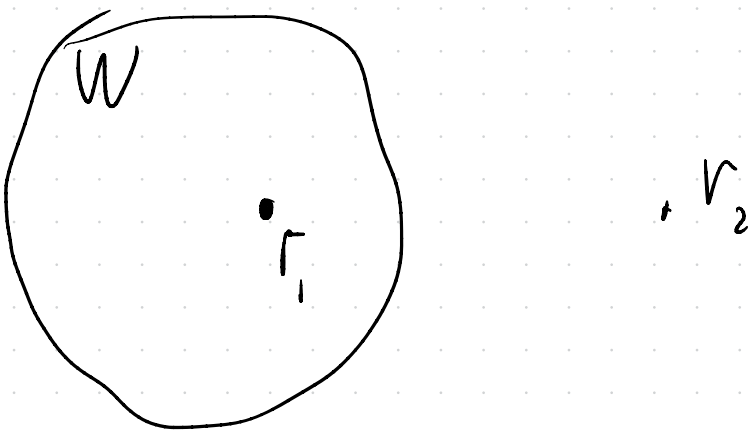
T for R in G

$C_f = \{f\}$

$C_c = \{e, c\}$

$C_a = \{a, b\}$

Say we pick $r_1, r_2 \in R$ & compute a min cut $\delta(W)$ separating them.



$$R_1 := R \cap W$$

$$R_2 := R \setminus R_1 = R \cap (V \setminus W)$$

Take $u, v \in R_1$. There is a (u,v)-min cut $\delta(X)$

s.t. $X \subseteq W$

GomoryHuAlg(G, R)

if $|R| = 1$,

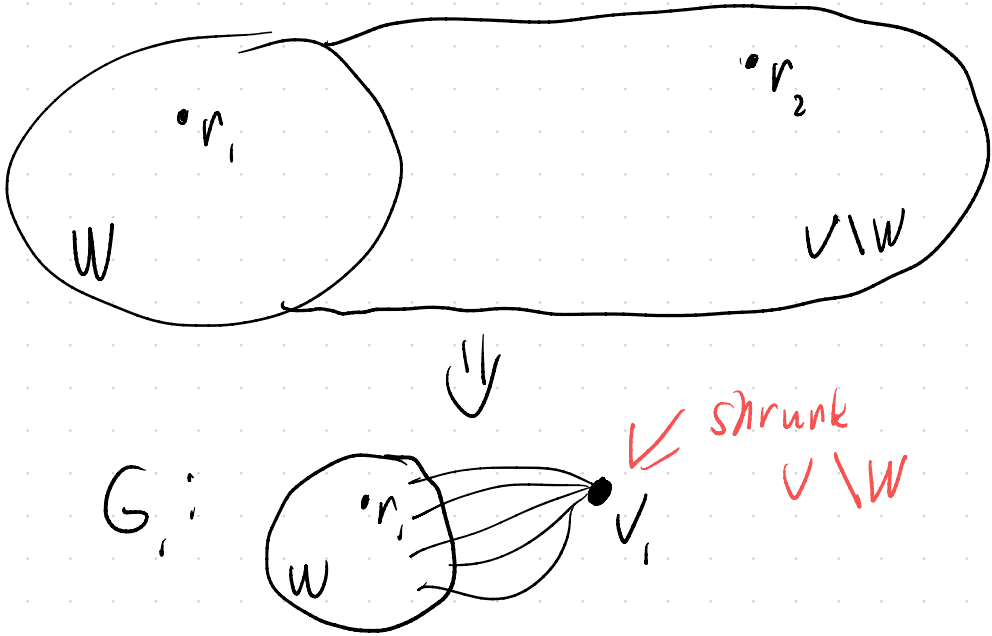
return $T = (\{r\}, \emptyset)$, $C_r = V$

else

let $r_1, r_2 \in R$, & let $\delta(W)$ be an (r_1, r_2)-min-cut

$G_1 \leftarrow G$ with $V \setminus W$ shrunk to a single vertex v_1

$R_1 \leftarrow R \cap W$



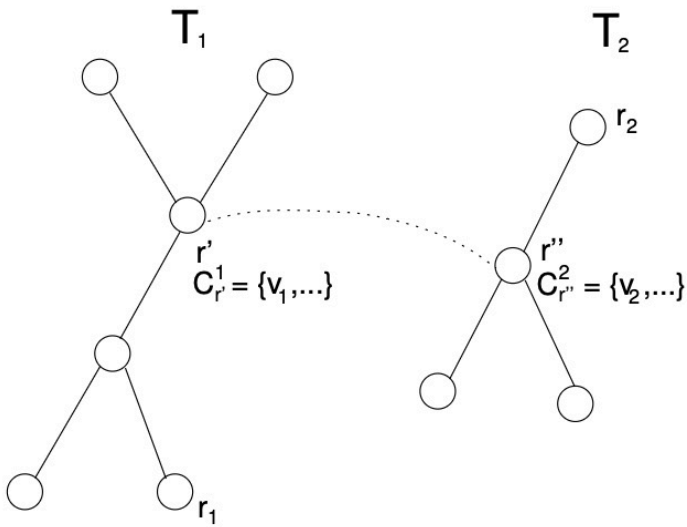
$G_2 \leftarrow G$ with W shrunk to a single vertex v_2

$R_2 \leftarrow R \setminus W$

-recurse-

$T_1(C^1_r \mid r \in R_1) = \text{Gomory Hu Alg}(G_1, R_1)$

$T_2(C^2_r \mid r \in R_2) = \text{Gomory Hu Alg}(G_2, R_2)$



let r' be the vertex in T_1 s.t. $v_1 \in C^1_{r'}$

let r'' be the vertex in T_2 s.t. $v_2 \in C^2_{r''}$

($r' \neq r''$ may not be $r_1 \neq r_2$!)

$T \leftarrow (R_1 \cup R_2, E_{T_1} \cup E_{T_2} \cup \{r', r''\})$

$(C_r \mid r \in R) \leftarrow$ ComputePartitions$(R_1, R_2$

$C'_r, C^2_r,$
$r', r'')$

return T, C_r

ComputePartitions $(R_1, R_2, C'_r, C^2_r, r', r'')$

for $r \in R_1, r \neq r', C_r \leftarrow C'_r$

for $r \in R_2, r \neq r'', C_r \leftarrow C^2_r$

$C_{r'} \leftarrow C'_{r'} - V_1$

$C_{r''} \leftarrow C^2_{r''} - V_2$

return $(C_r \mid r \in R)$

Thm: GomoryHuAlg returns tree for R.

Proof:

If $|R| = 1$, we're trivially fine.

o.w.

Let $st \in E_T$.

If $st \in T_1$ or $st \in T_2$, we're fine by induction & the fact that there is a non-crossing min (s,t)-cut.

Suppose then $st = r'r''$.

Cut $T - r'r''$ corresponds to

$\delta(W)$ in G.

So want to show $\delta(W)$ is a (r', r'')-min cut.

In particular, enough to show $\alpha_G(r', r'') \geq \alpha_G(r_1, r_2)$.

Suppose there is some (r', r'') cut $\delta(X)$ s.t. $c(\delta(X)) < c(\delta(W))$.

WLOG, $r' \in X$, $r'' \notin X$.

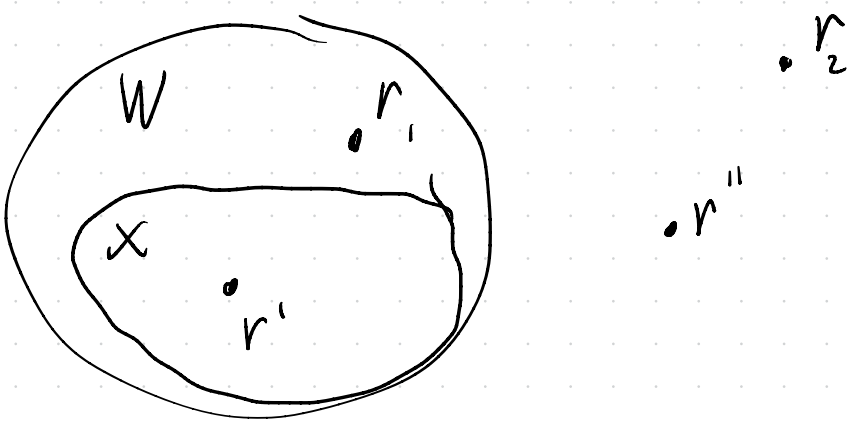
We have $r_1, r' \in W$.

X does not separate $r_1 + r_2$,

Because it is cheaper than min (r_1, r_2)-cut.

So both $r_1, r_2 \in X$ or neither are.

Suppose $r_1, r_2 \notin X$.

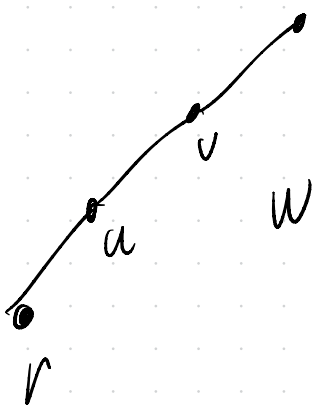


So $r' \in X$ & $r_1 \notin X$, so X separates r' from r_1.

So $\alpha_{G_1}(r', r_1) \leq c(\delta(X)) < c(\delta(W)) =$

$\alpha_G(r_1, r_2)$.

T_1



$w_1(uv) = \alpha_{G_1}(r', r_1)$

uv's cut separates r_1 from r_2 but is too cheap!

$\perp$

To compute tree for G, set $R \leftarrow V$.

Corollary: A Gomory-Hu for $R \subseteq V$ can be computed in $|R|-1$ min cut computations

$\Rightarrow O(n^2 m)$ time total

But there are faster algs!

All applies to non-neg. symmetric submodular functions!