Given a finite set N,
a real-valued set function
$f : 2^N \to \mathbb{R}$ is

submodular if for all
$A, B \in 2^N$,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

-or-

f is submodular if

$$f(A + v) - f(A) \geq f(B + v) - f(B)$$

for all $A \subseteq B \subseteq N \setminus B$.

Monotone submodular functions have $f(A) \leq f(B)$ for all $A \subseteq B$.

Non-negative " have $f(A) \geq 0$ $\forall A \subseteq N$

Symmetric have $f(A) = f(N \setminus A)$ for all $A \subseteq N$

Normalized if $f(\emptyset) = 0$.

Lemma: Let $G = (V, E)$ be an undirected graph with non-negative capacities $c : E \to \mathbb{R}_+$.

Let $f : 2^V \to \mathbb{R}_+$ be

$$f(A) = c(\delta(A))$$

↑ total capacity leaving a subset

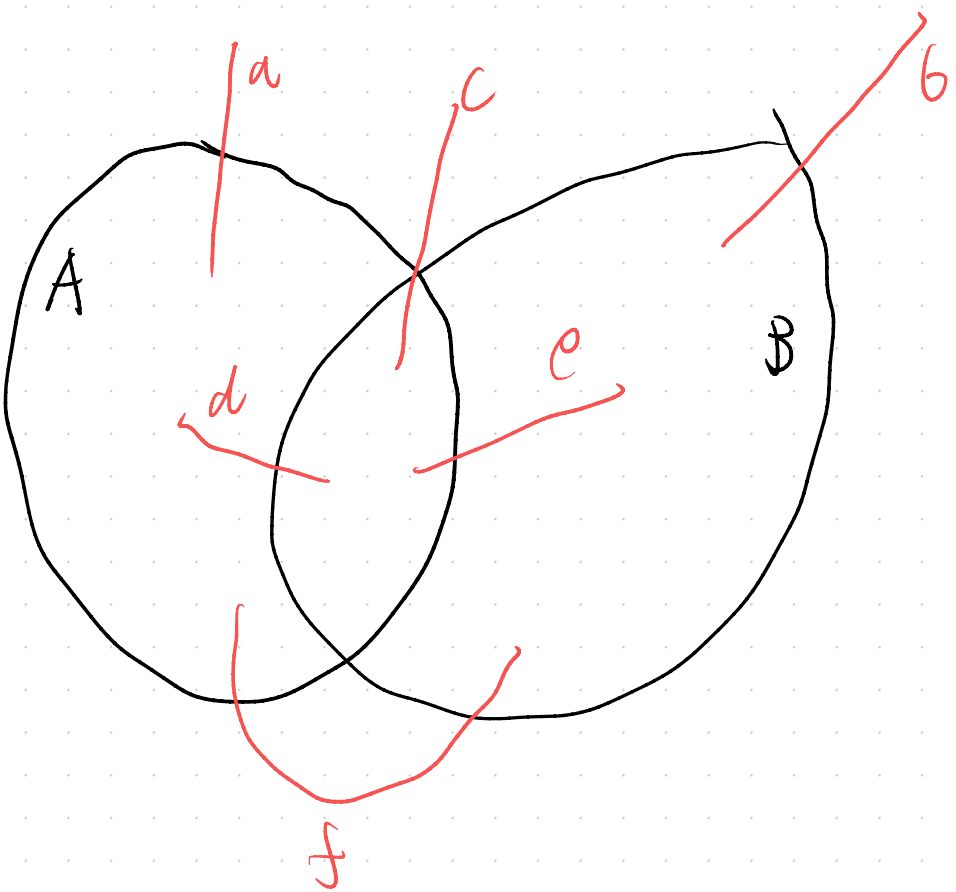
f is non-negative, symmetric,

normalized,

& submodular.

Proof:

$A, B \subseteq V$



a: capacity of edges ~~between~~

$(A - B)$ to $V - (A \cup B)$

$$f(A) + f(B) = a + c + e + f$$
$$+ b + c + d + f$$
$$= a + b + 2c + d + e + 2f$$

$$f(A \cup B) + f(A \cap B) = a + b + c$$
$$+ c + d + e$$
$$= a + b + 2c + d + e$$

So $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

$\Rightarrow$ submodular

$f: 2^N \to \mathbb{R}$ is called

posi-modular if

$$f(A) + f(B) \geq f(A-B) + f(B-A)$$

Lemma: If $f: 2^N \to \mathbb{R}$ is symmetric + submodular, then it is posi-modular.

Proof:

$$f(A) + f(B) = f(N-A) + f(B)$$ (symmetry)

$$\geq f((N-A) \cap B) + f((N-A) \cup B)$$ (submodular)

$$= f(B-A) + f(N-(A-B))$$

$= f(B-A) + f(A-B)$ ↖ symmetry

□

$\Rightarrow$ $c(\delta(A))$ is posi-modular

Typically when working with a submodular function, we assume access to a value oracle: a procedure/black box that given $A \subseteq N$, returns $f(A)$.

Running time typically measured

as # calls to oracle +
additional operations.

Thm: Given a submodular set function $f: 2^N \to \mathbb{R}$ via a value oracle, we can find $\min_{S \subseteq N} f(S)$ in strongly polynomial time.

$\Rightarrow$ Corollary: There is a strongly poly time algorithm that given a flow network $G = (V, A)$ & $s, t \in V$, outputs

$\alpha_G(s,t)$.

↑ capacity of min (s,t)-cut

Proof: Let $f: 2^{V-\{s,t\}} \to \mathbb{R}_+$

with $f(A) = c(\delta^+(A+s))$

↑ capacity leaving $A+s$

Want to minimize $f(A)$, &

we can show f is submodular.

Given symmetric set function

$f: 2^V \to \mathbb{R}_+$ & distinct $s, t \in V$.

Let $\alpha_f(s,t) = \min_{\substack{U \subset V \\ s \in U \\ t \in V - U}} f(U)$.

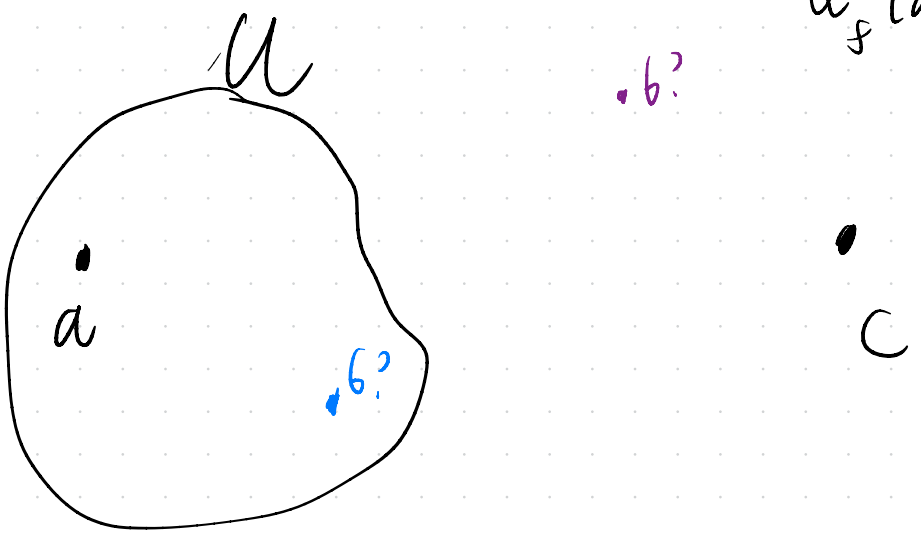
Lemma: For any symmetric

$f: 2^V \to \mathbb{R}_+$ & three elements

$a, b, c \in V$

$$\alpha_f(a,c) \geq \min\{\alpha_f(a,b), \alpha_f(b,c)\}$$

Proof: Consider min cut U separating a & c (i.e. $f(U) = \alpha_f(a,c)$)



- If $b \in U$, then
$$\alpha_f(b,c) \leq f(U) = \alpha_f(a,c)$$
- If $b \in V - U$, then
$$\alpha_f(a,b) \leq f(U) = \alpha_f(a,c)$$

Lemma can be used to show

$\leq n-1$ possible values

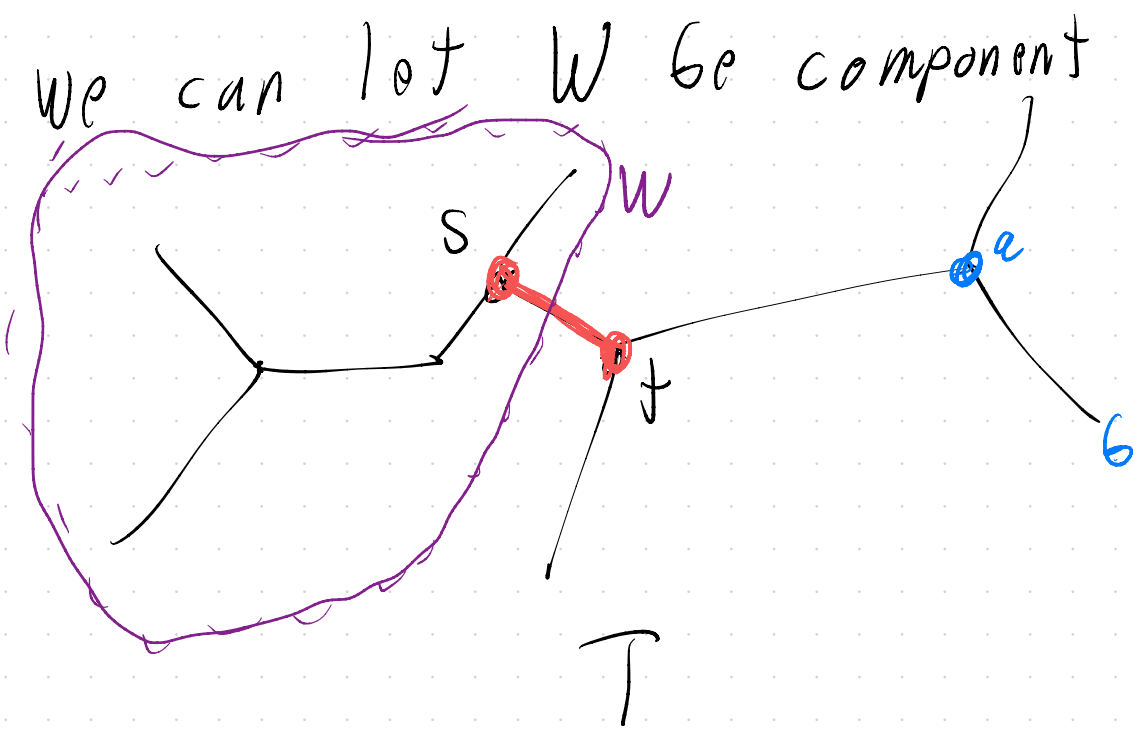
for $\alpha_f(\cdot,\cdot)$ if f is

symmetric.

But if f is submodular...

Let $G = (V, E)$ with capacities $c : E \to \mathbb{R}_+$.

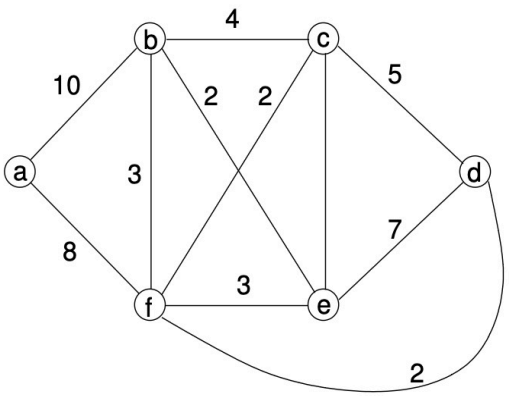
A tree $T = (V, E_T)$ is Gomory-Hu tree if for all $st \in E_T$,



of T-st & have W be a min (s,t)-cut.

Suppose a Gomory-Hu tree exists...

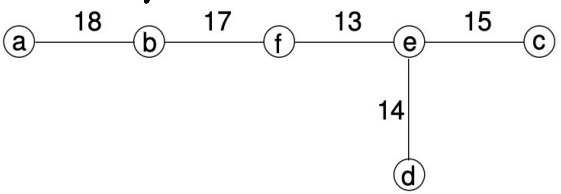
Label its edges with the capacity of the min cut it represents...



G

min (a,t)-cut?

$\alpha_G(a,t) = 17$ given by $W = \{a, b\}$



T

Thm: Let T be a Gomory-Hu tree for graph $G = (V, E)$. For all $u, v \in V$ distinct vertices, let st be the edge on the unique (u,v)-path in T minimizing $\alpha_G(s,t)$, then

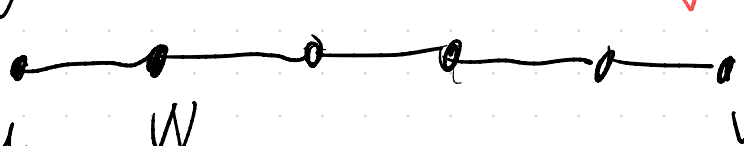
$$\alpha_G(u,v) = \alpha_G(s,t).$$

the cut W from $T-st$ is a min (u,v)-cut.

$\Rightarrow$ with those labels of T, $\alpha_G(u,v) = \alpha_T(u,v)$ for all $u,v \in V$.

Proof: Consider path from u to v in T. If $uv = st$, then $\alpha_G(u,v) = \alpha_G(s,t)$ trivially.

Otherwise, let w be (u,v)-path adjacent to u on the path in T



$\alpha_G(u,v) \geq \min\{\alpha_G(u,w), \alpha_G(w,v)\}$.

If $uw = st$, then $\alpha_G(u,v) = \alpha_G(s,t)$.

o.w. by induction on path length, $\alpha_G(u,v) \geq \alpha_G(w,v)$

$\geq \alpha_G(s,t)$.

Any edge (on the path) including st gives a cut separating $u \neq v$, so $\alpha_G(u,v) \leq \alpha_G(s,t)$.

Together $\alpha_G(u,v) = \alpha_G(s,t)$.