Given directed $G = (V, A)$,

capacities $c : A \to \mathbb{R}_+$,

$s, t \in V$.

flow on arc a

conservation at vertices

$$\max \sum_{a \in \delta^+(s)} x(a) - \sum_{a \in \delta^-(s)} x(a)$$

$$\sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = 0 \quad \forall v \in V - \{s,t\}$$

$$x(a) \leq c(a) \quad \forall a \in A \quad \leftarrow \text{capacity constraints}$$

$$x(a) \geq 0 \quad \forall a \in A$$

max flow problem

Constraint matrix is a TUM

$\Rightarrow$ if c is integral, there is an integral optimal

solution

Dualizes to

$$\min \sum_{a \in A} c(a)y(a)$$

$$\begin{aligned}
z(u) - z(v) + y(u,v) &\geq 0 \quad (u,v) \in A \quad \{u,v\} \cap \{s,t\} = \emptyset \\
-z(v) + y(s,v) &\geq 1 \quad \forall (s,v) \in A \\
z(v) + y(s,v) &\geq -1 \quad \forall (v,s) \in A \\
z(v) + y(v,t) &\geq 0 \quad \forall (v,t) \in A \\
-z(v) + y(t,v) &\geq 0 \quad \forall (t,v) \in A \\
y &\geq 0
\end{aligned}$$

Theorem: Maximum-flow minimum theorem (variant):

Max flow value = min total capacity of a set of arcs whose removal separates s from t.

Let y^*, z^* be optimal for our dual.

Let $A' = \{a \in A \mid y^*(a) \geq 1\}$.

$$c(A') := \sum_{a \in A'} c(a) \leq \sum_{a \in A} c(a) y^*(a).$$

Constraints in a TUM rhs integral

$\Rightarrow$ can assume z^*, y^* integral

Need to argue A' separates s from t.

Suppose otherwise.

So there is a path

$$P = s, v_1, v_2 \dots, v_k, t \text{ in } G - A'.$$

(Assuming no edge (s,t)
$\Rightarrow k \geq 1$)

y^* integral, so all arcs a in P have $y^*(a) = 0$.

By second constraint $z^*(v_1) \leq -1$.

By first, $z^*(v_{i+1}) \leq z^*(v_i)$ for $i \leq k-1$.

So, $z^*(v_k) \leq -1$.

But by fourth constraint,

$z(v_k) \geq 0$. ✓

Can interpret y^* & z^*.

Can treat y^* as edge lengths.

$z^*(v)$: (distance from s to v) -1

Need (distance to s) $-1 = -1$

(distance to t) $-1 = 0$

A <u>circulation</u> in $D = (V, A)$ is a function $f : A \to \mathbb{R}_+$ s.t.

$$\sum_{a \in \delta^-(v)} f(a) = \sum_{a \in \delta^+(v)} f(a)$$

$\forall v \in V$

Suppose given $\ell : A \to \mathbb{R}_+$

$u : A \to \mathbb{R}_+$.

Can we find a circulation f s.t. $\ell(a) \le f(a) \le u(a)$ forall $a \in A$?

Can express using another

CP instance.

Hoffman

Thm: There is a feasible circulation iff

1) $\ell(a) \le u(a) \quad \forall a \in A$ &

2) $\forall U \subseteq V, \ell(\delta^-(U)) \le u(\delta^+(U))$

total bound on all arcs leaving (-) or entering (+) U

Algorithms for Max Flow:

Given flow network

$G = (V, A), c, s, t, d$

an (s,t)-flow $f: A \Rightarrow \mathbb{R}_+$,

there is a <u>residual graph</u>

$G_f(V, A')$:

(Assuming we don't have both

(u,v) + (v,u) in A).

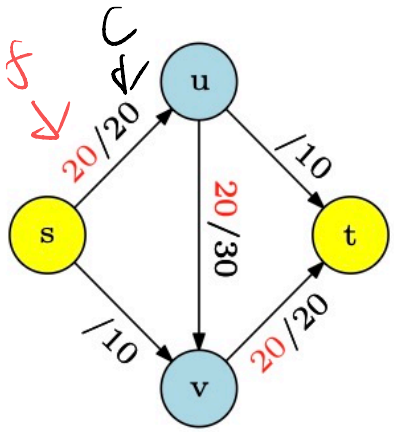
For each arc $(u,v) \in A$:

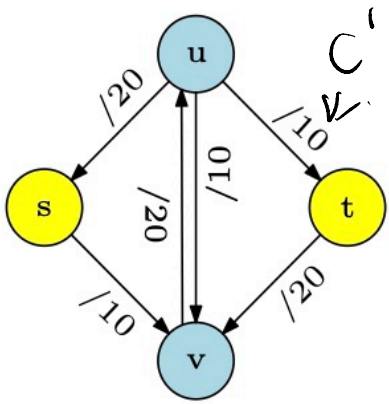
If $0 \leq f(a) < c(a)$, add (u, v) to A' with residual capacity $c'(u,v) = c(u,v) - f(u,v)$

& add (v, u) to A' with res. capacity of $c'(v,u) = f(u,v)$

If $f(u,v) = c(u,v)$ ((u,v) is saturated),

add only (v, u) to A' with res. capacity of $f(u,v) = c(u,v)$





If $f(u,v) = 0$, add (u,v) to A' with $c'(u,v) = c(u,v)$

Given $f : V \times V \to \mathbb{R}^+$

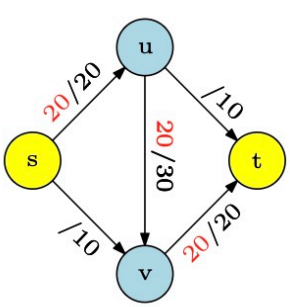
$g : V \times V \to \mathbb{R}$

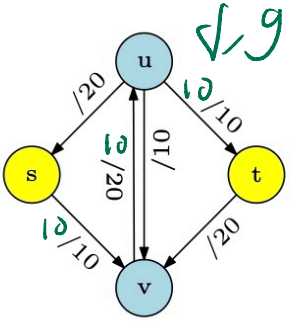
$f+g\,(u,v) = f(u,v) + g(u,v) - f(v,u)$

Lemma: Given flow f in G & flow g in G_f.

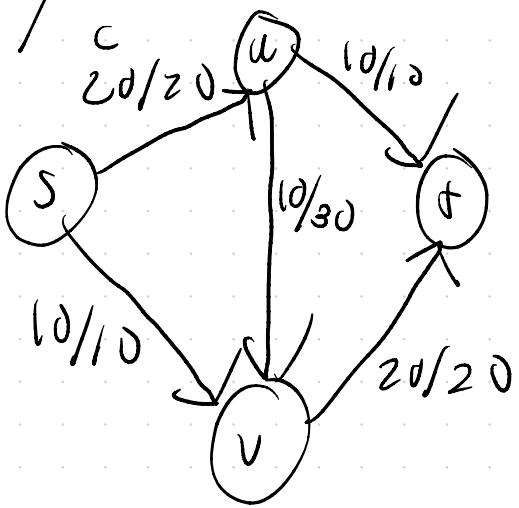
Then $f+g$ is an (s,t)-flow in G of value $|f|+|g|$.

$$(f+g)(u,v) = f(u,v) + g(u,v) - g(v,u)$$





$(f+g)$ /



Works for any (s,t)-flow g in G_f.

Lemma: Let f & g be flows in G. Then $g \cdot f$ is an (s,t)-flow of G_f of value $|g| - |f|$.

if $g(u,v) - f(u,v) \geq 0$

set $(g-f)(u,v) =$

$g(u,v) - f(u,v)$

o.w. set $(g \cdot f)(v,u) =$

$f(u,v) - g(u,v)$

Corollary: Flow f is max value iff no (s,t) path in G_f.

(Recall, all edges in G_f have positive capacity)

Ford-Falkerson Augmenting (Paths):

Start with the all-0 flow

While there is a positive value flow g in G_f:

$f \leftarrow f + g$

If c is integral, then max flow f^* is integral.

Can always pick integral g s.t. $|g| \geq 1$.

So $\leq |f^*|$ iterations.

a m) time per iteration

so $O(m/\delta^*)$ time

pseudo polynomial time

poly in # magnitude, but exponential in input size