A graph $G = (V, E)$ on n nodes is an *interval graph* if there is a collection I of n intervals on the real line

there is a bijection $f : V \to I$

(each vertex corresponds to an interval)

$\forall$ $uv \in E$ if $f(u)$ & $f(v)$ intersect.

Problem: Given a weight

$w : V \to \mathbb{R}^+$.

Want a max weight independent set in G i.e. a max weight set of vertices s.t. no two in the set share an edge.

(i.e. find a max weight set of intervals that don't overlap)

Let $\mathcal{I} = \{I_1, \ldots, I_n\}$

$$\max \sum_{i=1}^{n} w_i x_i$$

w_i ← weight of I_i

x_i ← do we take I_i

$$\sum_{I_i : p \in I_i} x_i \leq 1 \quad \forall p \in \mathbb{R}$$

$\forall p$ an endpoint of an interval

$$x_i \geq 0 \quad \forall \; 1 \leq i \leq n$$

Sort endpoints from left to right & write constraints in that order.

It's a consecutive 1s matrix

$\Rightarrow$ matrix is a TUM

$\Rightarrow$ polyhedron is integer

$\Rightarrow$ optimal solution to LP is integral (and 0,1)

So... solve LP & take intervals that got a 1.

Integer Decomposition Property:

Polyhedron P has the integer decomposition property if for all integers $k \geq 1$ and $x \in P$,

kx is integral implies $kx = x_1 + x_2 + \cdots + x_k$ for integral vectors $x_1, \ldots, x_k \in P$.

Thm: Matrix A is TUM iff $P = \{x \mid x \geq 0, Ax \in b\}$ has the

integer decomposition property for all integral vectors b.

Claim: Suppose we have intervals I s.t. $\forall p \in \mathbb{R}$, at most k intervals contain p.

I can be partitioned into $I_1, I_2, \ldots, I_k$ such that each I_j is a collection of disjoint intervals.

i.e. interval graph has a k coloring

call is vector

Proof: $x^* = \frac{1}{k} \cdot \bar{I}$ is in the ind. set polyhedron.

$y = k x^* = \bar{I}$ is integral

$y = x_1 + \dots + x_k$ where each x_i is integral & in P.

$I_i := x_i$ to get the partion.

Network Flow:

Let $G = (V, A)$ be directed.

Let $c : A \to \mathbb{R}_+$ be the arc capacities.

Let $s, t \in V$

flow network

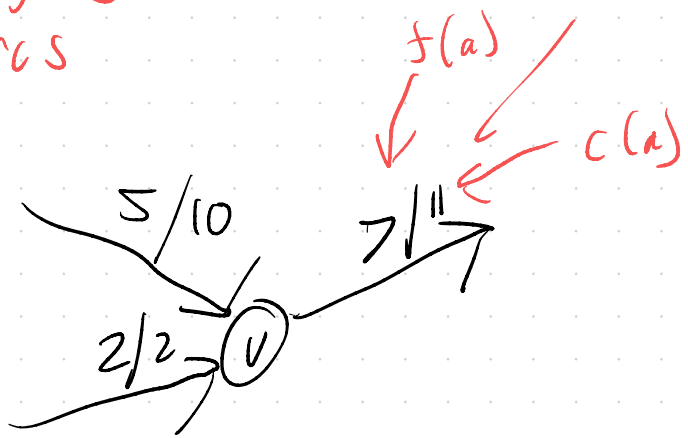
A real valued $f : A \to \mathbb{R}$ is an (s, t)-flow if it satisfies these conditions:

- flow conservation at vertices

for all nodes $v \notin \{s, t\}$

$$\sum_{a \in \delta^+(v)} f(a) = \sum_{a \in \delta^-(v)} f(a)$$

↑ outgoing arcs



- capacity constraints on all arcs: For each $a \in A$, $0 \leq f(a) \leq c(a)$

The value of flow f is

$$val(f) = |f| := \sum_{a \in \delta^+(s)} f(a) - \sum_{a \in \delta^-(s)} f(a)$$

↑ net amount leaving s

$$\left(= \sum_{a \in \delta^-(t)} f(a) - \sum_{a \in \delta^+(t)} f(a) \right)$$

Max (s,t)-Flow: Find an (s,t)-flow of max value.

As an LP: case $x(a) = f(a)$

$$\max \sum_{a \in \delta^+(s)} x(a) - \sum_{a \in \delta^-(s)} x(a)$$

$$\sum_{a \in \delta^+(v)} x(a) - \sum_{a \in \delta^-(v)} x(a) = 0 \quad \forall v \in V - \{s,t\}$$

$$x(a) \leq c(a) \quad \forall a \in A$$

$$x(a) \geq 0 \quad \forall a \in A$$

Polyhedron is of form $\{x \mid Mx = 0$
$0 \leq x \leq c\}$

M' is a submatrix of the vertex-arc incidence matrix.

$\Rightarrow M'$ is a TUM

If c is integral, polyhedron is integer & its optimal solutions are integral.

(i.e., There is an integral max value flow if capacities are integral)

Dual LP

Variables $y(a)$ $a \in A$

$z(v)$ $v \in \{s,t\}$

$$\min \sum_{a \in A} c(a)\, y(a)$$

$$z(u) - z(v) + y(u,v) \geq 0 \quad \forall (u,v) \in A$$

$$\{u,v\} \cap \{s,t\} = \emptyset$$

$$-z(v) + y(s,v) \geq 1 \quad \forall (s,v) \in A$$

$$z(v) + y(s,v) \geq -1 \quad \forall (v,s) \in A$$

$$z(v) + y(v,t) \geq 0 \quad \forall (v,t) \in A$$

$$-z(v) + y(t,v) \geq 0 \quad \forall (t,v) \in A$$

$y \geq 0$

There is an integral optimal solution.

y^*, z^*: optimal solution

Let $A' = \{a \in A \mid y^*(a) \geq 1\}$

$$c(A') := \sum_{a \in A'} c(a) \leq \sum_{a \in A} c(a) y^*(a)$$

Claim: No (s,t)-path avoids every arc in A'.

$\Rightarrow$ max-flow value = min cut capacity