A is totally unimodular (is a TUM) if every submatrix has a determinant in $\{0, 1, -1\}$.

Theorem: Let A be a matrix with entries in $\{0, 1, -1\}$. The following are equivalent:

1) A is TUM
2) For all integral vectors b, $\{x \mid Ax \leq b, x \geq 0\}$ is an integer polyhedron.

3) For all integer a, b, c, d,

$\{x \mid a \leq x \leq b, c \leq Ax \leq d\}$

is an integer polyhedra.

4) Each collection of columns S from A can be split into two sets S_1 & S_2 such that the sum of columns in S_1 minus the sum from S_2 gives a vector with entries in $\{0, 1, -1\}$.

5) Each nonsingular submatrix of A has a row with an odd # nonzero components.

6) No square submatrix of A has determinant 2 or -2.

Can solve any TUM LP instance in strongly poly time (Tardös).

However, there are usually problem-specific combinatorial algorithms that are faster.

Examples:

Bipartite graphs

Let $G=(V,E)$ be undirected.

Let M be the $\{0,1\}$ edge-vertex incidence matrix.

$|E| \times |V|$

$M_{e,v} = 1$ iff e is incident to v.

(e has v as an endpoint)

Claim: M is a TUM iff G is bipartite.

Directed graphs:

Let $D = (V, A)$ be directed.

Let M be an $|A| \times |V|$ arc-vertex incidence matrix

$$M_{a,v} = \begin{cases} 0 & \text{if } a \text{ not incident to } v \\ 1 & \text{if } a \text{ leaves } v \\ -1 & \text{if } a \text{ enters } v \end{cases}$$

Poincaré (1900): M is always a TUM.

Consecutive 1s matrix

A is a consecutive 1s matrix if all entries in $\{0, 1\}$, and in each row, all the 1s form a consecutive block.

Always TUM.

(Can do 1 blocks for columns instead.)

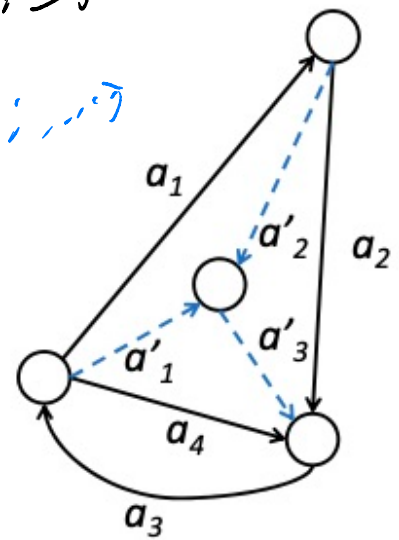
A network matrix M is defined for a directed graph $D = (V, A)$ & directed tree $T = (V, A')$ on the same vertices.

M is $|A'| \times |A|$

$$M_{a', u \to v} = \begin{cases} 0 & \text{if unique } (u,v)\text{-path in } T \text{ does not use } a' \\ 1 & \text{if path uses } a' \text{ in forward direction} \\ -1 & \text{if path uses } a' \text{ in backward direction} \end{cases}$$

A: →

A': ⋯→



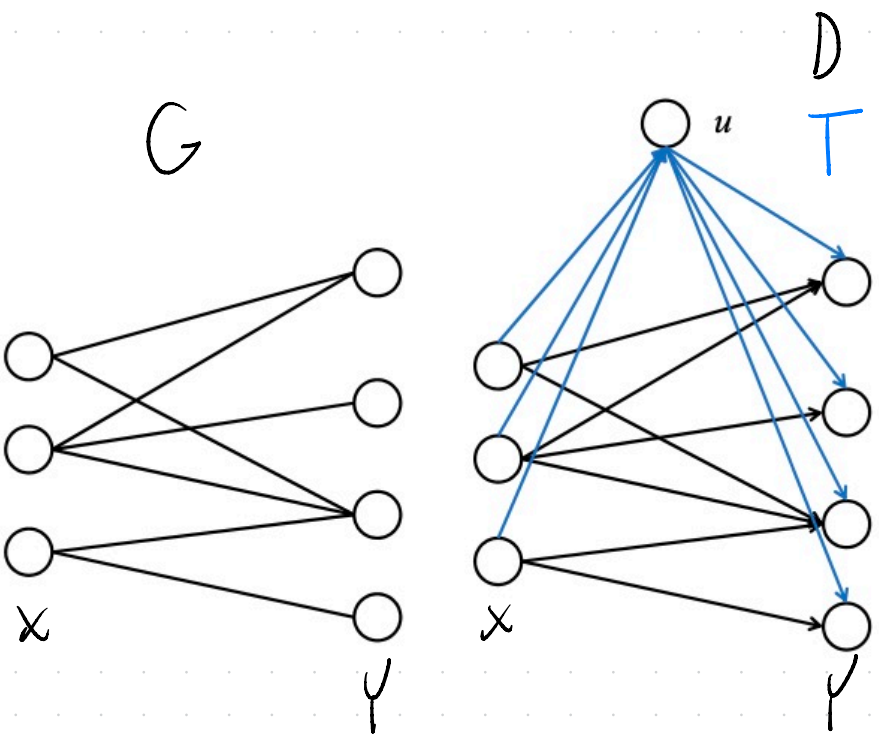
$$M = \begin{array}{c} \\ a'_1 \\ a'_2 \\ a'_3 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

Bipartite graphs: Say $G = (X \sqcup Y, E)$.

Let $D = (X \cup Y \cup \{u\}, A)$ where u is new & A is all edges of G oriented from X to Y.

Let $T = (X \cup Y \cup \{u\}, A')$

$A' = \{(v,u) \mid v \in X\} \cup \{(u,v) \mid v \in Y\}$



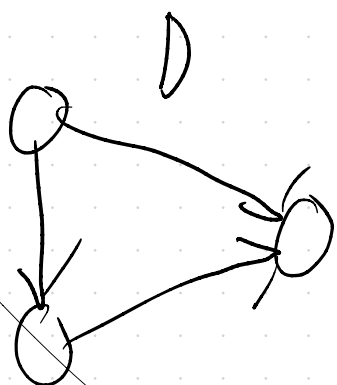
Directed graphs:

Suppose $D = (V, A)$ is directed.

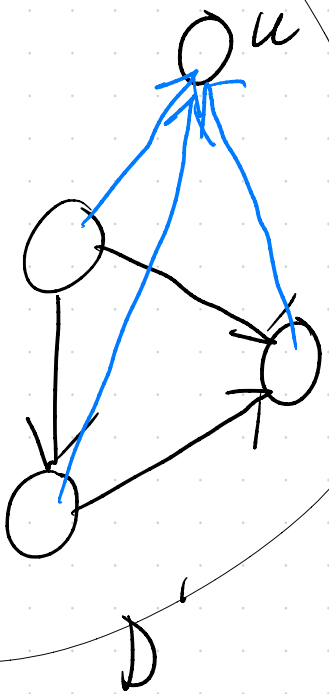
Create $D' = (V \cup \{u\}, A)$ &

$T = (V \cup \{u\}, A')$

$A' = \{(v, u) \mid v \in V\}$



=>



Consecutive 1s.

Suppose M is a consecutive $m \times n$ 1s matrix.

Assume each row has at least

one 1.

ℓ_i : leftmost column to hold 1 in row i

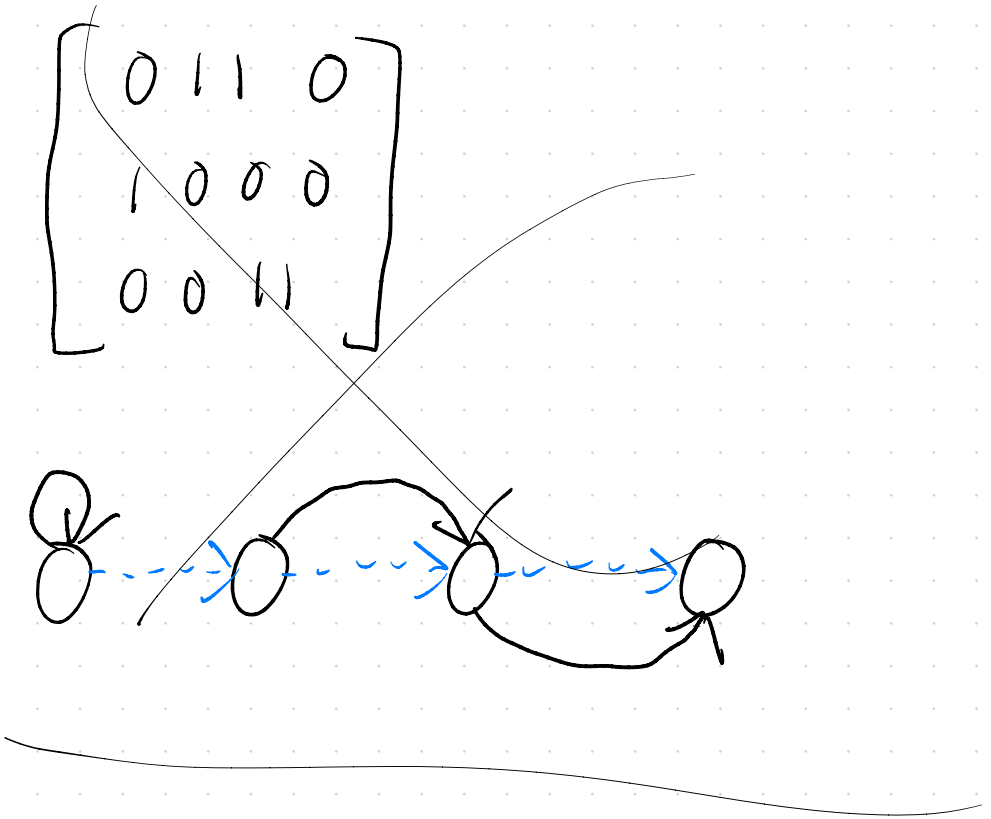
r_i : rightmost column to hold 1 in row i

$V := \{1, 2, \ldots, n\}$. Need to double check this :

$T = (V, A')$

$A' = \{(j, j+1) \mid 1 \leq j < n\}$

$D = (V, A)$ $A = \{(\ell_i, r_i) \mid 1 \leq i \leq m\}$



Applications:

Bipartite graph matching:

Let $G = (V, E)$ be bipartite with $V = V_1 \sqcup V_2$.

max cardinality matching:

$$\max \sum_{e \in E} x(e)$$

$e \in E$ — incident edges

$$x(\delta(u)) \leq 1 \quad \forall u \in V$$

$$x(e) \geq 0 \quad \forall e \in E$$

$$x \in \mathbb{Z}$$

i.e. $\max \{ 1 \cdot x \mid Mx \leq 1,\ x \geq 0,\ x \in \mathbb{Z} \}$

M — vertex-edge incidence matrix

M is a TUM, so

$\{ Mx \leq 1, x \geq 0 \}$ is a integer polyhedron.

Could just solve LP to get max matching.

Dual is

$$\min \sum_{u \in V} y(u)$$

$$y(u) + y(v) \geq 1 \quad \forall uv \in E$$

$$y \geq 0$$

$\{y^T M \geq 1, y \geq 0\}$ is an integer polyhedron, because M^T is a TUM.

Optimal solution is integral

$\{0, 1\}$

i.e. select a subset of vertices

such that for each $uv \in E$,
at least one of u or v
in subset.

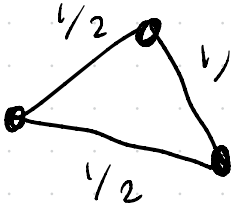
i.e. we want a min cardinality
vertex cover.

By strong duality, optimum is
same for both.

Theorem (König). In a bipartite
graph, the cardinality of a max
matching is equal to that of
a min vertex cover.

& can find them in polynomial
time.

What if G not bipartite?



$Mx \leq 1, x \geq 0$ not TUM!

So solving (this) LP does not help!

Can weight edges & ask for

$$\max \sum_{e \in E} w(e) \cdot x(e).$$

Want max weight matching.

Now dual constraints are

$$y(u) + y(v) \geq w(uv)$$

but still want $\min \sum y(u)$

Want a "min w-cover".

So for integer w,

max matching weight =

min size w-cover (can take some vertices multiple times)

in bipartite graphs

König-Egervary theorem