An integer (linear) programming (ILP) problem has

a - linear objective function
- linear constraints
- solutions required to be an integer vector

$$\max \; c^T x$$

$$Ax \leq b$$

$$x \in \mathbb{Z}^n$$

NP-hard to solve in general

probably no poly time algorithm

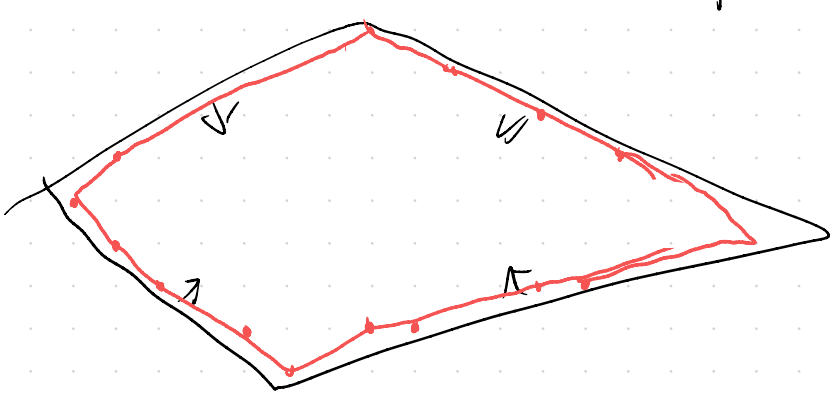
hard even to tell if there is any solution!

Lenstra '(83): poly time algorithm for when n is a fixed constant

Integer polyhedra:

Given a rational or real polyhedron $P = \{x \mid Ax \leq b\}$, then P_I is the convex hull of integer vectors in P.

P_I: integer hull of P.



Thm: For any rational polyhedron P interer hull P_I is a polyhedron.

P is an integer polyhedron if and only if $P = P_I$.

Thm: The following are equivalent.

1. $P = P_I$
2. Every face of P contains an integer vector.
3. Every minimal face of P

contains an integer vector.

4. $\max\{c^T x \mid x \in P\}$ is attained by an integer vector whenever the optimum value is finite.

Proof:

1. $\Rightarrow$ 2.: Let F be a face of P, meaning $F = P \cap H$ for some supporting hyperplane H.

Let $x \in F$.

Because $P = P_I$, x is the convex combo of integer points in

P.

All points forming the convex combo lie on H.

Meaning these integer points lie in F as well.

2. $\Rightarrow$ 3. Just a special case.

3. $\Rightarrow$ 4.

Let $\delta = \max\{c^T x \mid x \in P\}$ be finite.

$F = \{x \in P \mid c^T x = \delta\}$ is a face of P.

Any minimal face of F has an integer vector attaining δ.

4. $\Rightarrow$ 1.

Suppose for contradiction there is some $y \in P \setminus P_I$.

Then there is an inequality $ax \leq \beta$ valid for P_I while $ay > \beta$.

Therefore $\max\{ax \mid x \in P_I\} \leq \beta$ but $\max\{ax \mid x \in P\} > \beta$ because of y, violating 4. for $c^T = a$. $\square$

Corollary: A pointed polyhedron P is an integer polyhedron iff each vertex of P is an integer vector.

Edmonds & Giles: A rational polyhedron P is integral iff $\max\{c^T x \mid x \in P\}$ is an integer for each integer vector c for which the max is finite.

Combinatorial optimization:

Can typically model combinatorial optimization problems as ILP problems in a natural fashion.

Suppose we have an ILP of the form $\max \{ c^T x \mid Ax \leq b, x \in \mathbb{Z}^n \}$.

If lucky, the underlying LP $\max \{ c^T x \mid Ax \leq b \}$ uses an integer polyhedron & the

number of constraints is polynomial in size of problem we're modeling.

Then call a poly time LP solver to solve ILP also.

Sometimes can solve in poly time even if an exponential # constraints using ellipsoid method.

Matrix A is totally unimodular (a TUM) if the determinant of each square submatrix of A is in $\{0, 1, -1\}$.

$\Rightarrow$ Every entry is in $\{0, 1, -1\}$.

Claim: If A is a TUM & U is a non-singular square submatrix of A, then U^{-1} is a matrix with entries in $\{0, 1, -1\}$.

Proof:

$$U^{-1} = \frac{adj(U)}{det(U)}$$

$adj(U)$: adjoint matrix of U

$adj(U) = C^T$ where C is the cofactor matrix of U

$C_{i,j}$: determinant of sub-matrix of U obtained by removing row i column j.

$C_{i,j} \in \{0, 1, -1\}$ $\quad det(U) \in \{1, -1\}$

Thm: If A is a TUM, then for all integer vectors b, the polyhedron $P = \{x \mid Ax \leq b\}$ is an integer polyhedron.

Proof:

Suppose P is non-empty and any (non-empty) minimal face F.

$F = \{x \mid A'x = b'\}$ for some subsystem $A'x \leq b'$ of $Ax \leq b$ where A' has full row rank.

$\Rightarrow A'$ has $m' \leq n$ rows.

A' can be written as

$A' = [U \quad V]$ where U is a $m' \times m'$ square matrix of full row & column rank.

$\Rightarrow U$ is invertible

Every entry in U^{-1} is an integer by previous Claim.

Let y be a solution to $A'x = b'$

We have $y = U^{-1}b$, making it

an integer vector.

In particular, every minimal face of P has an integer vector $\Rightarrow$ P is an integer polyhedron.

Claim: A is a TUM iff any A' obtained by multiplying any row or column by -1 is a TUM.

Corollary: If A is a TUM, then for all integer vectors a, b, c, d, the polyhedron

$$\{x \mid a \leq x \leq b,\ c \leq Ax \leq d\}$$

is an integer polyhedron.

called <u>box constraints</u>

"Proof": Suppose A is a TUM.

Then the matrix

$$A' = \begin{bmatrix} I \\ -I \\ A \\ -A \end{bmatrix}$$ is also TUM.

Claim: A is a TUM

iff

A^T is a TUM.

Corollary: Suppose A is a TUM & b & c are integer vectors.

Then $\max\{c^T x \mid Ax \leq b, x \geq 0\}$

$= \min\{y^T b \mid y^T A \geq c, y \geq 0\}$

are attained by integer vectors x^* & y^* if the optima are finite

Proof: The polyhedron

$\{y \mid y \geq 0, y^T A \leq c^T\}$ is integer, because A^T is a TUM so is $\begin{bmatrix} A^T \\ -I \end{bmatrix}$.

So both primal & dual LPs are integer polyhedron & have integer optimal values.