CS 4301 — Special Topics in Computer Science: Optimization, Homework 2

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Primal	Dual	Primal	Dual
$\max c \cdot x$	$\min y \cdot b$	$\min c \cdot x$	$\max y \cdot b$
$\sum_j a_{ij} x_j \leq b_i$	$y_i \geq 0$	$\sum_j a_{ij} x_j \leq b_i$	$y_i \leq 0$
$\sum_j a_{ij} x_j \geq b_i$	$y_i \leq 0$	$\sum_j a_{ij} x_j \geq b_i$	$y_i \geq 0$
$\sum_j a_{ij} x_j = b_i$	—	$\sum_j a_{ij} x_j = b_i$	—
$x_j \geq 0$	$\sum_i y_i a_{ij} \geq c_j$	$x_j \leq 0$	$\sum_i y_i a_{ij} \geq c_j$
$x_j \leq 0$	$\sum_i y_i a_{ij} \leq c_j$	$x_j \geq 0$	$\sum_i y_i a_{ij} \leq c_j$
—	$\sum_i y_i a_{ij} = c_j$	—	$\sum_i y_i a_{ij} = c_j$
$x_j = 0$		$x_j = 0$	

1. Let $A \in R^{m \times n}$ and $b \in R^m$. In class, we saw that the primal LP instance

$$\max c^T x$$

$$Ax \leq b$$

has as its dual

$$\min y^T b$$

$$y^T A = c$$

$$y \geq 0$$

Prove that the dual of the dual LP is the primal LP. Do not simply rely on the chart above. Instead, explain the transformations you perform in order to use duality as defined in this problem.

Hint: Write an equivalent instance to the dual LP that it looks like the primal's standard form. To do so, add pairs of inequalities that together form the equality constraints and multiply the objective by −1 so it becomes a maximization. Then, verify the dual of your rewritten instance is equivalent to the primal LP.

First start with the dual LP. In the dual

$$\min y^T b$$
$$y^T A = c$$
$$y_i \geq 0$$

To turn the inequalities into equalities simply convert each $\sum_i y_i \cdot a_{i,j} = c_i$ into $\sum_i y_i \cdot a_{i,j} \leq c_i$ and $-\sum_j y_i \cdot a_{i,j} \leq -c_i$. For the objective, minimizing $y^T b$ is the same as maximizing $-y^T b$ (i.e $\max(-y^T b)$).

Dualizing this, we get

$$\min -c^T x$$
$$\sum_j x_i \cdot c_{i,j} \leq b$$
$$-\sum_j x_i \cdot c_{i,j} \leq -b$$
$$x_i \geq 0$$

which can be rewritten as

$$\max c^T x$$
$$Ax_i = b$$
$$x_i \geq 0$$

which is a standard form for the primal LP (i.e. they are equivalent)

Let $G = (V, E)$ be a bipartite graph with no isolated (degree-0) vertices. An edge cover of G is a number $C \subseteq E$ of edges such that each vertex in G is incident to at least one edge of C. The following integer programming instance captures the minimum cardinality edge cover problem in G.

$$\min \sum_{e \in E} x(e)$$

$$x(\delta(u)) \geq 1 \forall u \in V$$

$$x(e) \geq 0 \forall e \in E$$

$$x \in \mathbb{Z}$$

Recall from our discussion on bipartite matching that constraint matrix above is a TUM and therefore an optimal solution to the above integer program is also an optimal solution to the following *linear* programming instance.

$$\min \sum_{e \in E} x(e)$$

$$x(\delta(u)) \geq 1 \forall u \in V$$

$$x(e) \geq 0 \forall e \in E$$

Using the chart on the first page, derive the dual of the linear programming instance above

$$\min 1 \cdot x$$
$$x(\delta(u)) \geq 1$$
$$x \geq 0$$

$$\min 1 \cdot x$$
$$Mx \geq 1$$

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$$x \geq 0$$

for vertex-incidence matrix M we want

$$\max y \cdot 1$$
$$yM \leq 1$$
$$y \geq 0$$

———

$$\max \sum y(u)$$
$$y(u) + y(v) \leq 1 \forall (u, v) \in E$$

since each column of M represents an edge of the graph, and

$$y(u) \geq 0 \forall u \in V$$

since you must have 0 or more of each vertex.

An independent set in G is a subset $S \subseteq V$ of vertices such that no two vertices in S share an edge. What problem does an integral solution to the dual LP solve? (Again, the constraint matrix is a TUM, so there will always be some optimal solution that is integral.)

The dual LP finds the maximum independent set for the graph. This is because we are maximizing the number of vertices selected, each of the endpoints of each edge can only be used once (i.e. there are no adjacent vertices).

Use strong duality and the above observations to explain the relationship between edge cover sizes and independent set sizes in bipartite graphs.

By strong duality, the solution to the edge cover size (i.e. the number of edges taken in the minimum edge cover) and the independent set size (the number of vertices in the maximal independent set) are the same. According to strong duality, these two objective values should be the same (and as expected, these are our objectives).

So far, we have assumed G has no isolated vertices. Unfortunately, the strong duality argument above no longer applies if there is even a single isolated vertex, because neither the primal nor dual LP have a finite optimal solution. Concisely explain why the primal LP has no finite optimal solution by referencing only the primal LP. Similarly, explain why the dual LP has no finite optimal solution by referencing only the dual LP.

Primal

For the primal, if there are any isolated vertices then we must argue that the LP is empty. Because for that vertex $\delta(u) = 0$, the Integer Programming instance $x(\delta(u)) = 0$, which violates the second set of constraints. This means that no solution to the LP contains the optimal solution, which implies that the LP does not contain a finite optimal solution.

Dual

For the dual, if there are any isolated vertices, then $y(u)$ can be increased indefinitely for an isolated vertex $u \in V$, since it is not involved in the second inequality. This implies that the LP has no finite optimal solution, since it is unbounded.