

Algorithmic Game Theory

CS 58100 · Mechanism Design I

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Fall 2026

ANNOUNCEMENT

- No OH today
  - Prof. Makur's talk!

MECHANISM DESIGN

- Game Theory: Interaction of rational, competing, strategic agents
- Mechanism Design: “Inverse Game Theory”
  - How do we design systems for rational, competing, strategic agents?
  - We’ll be interested in promoting a desired objective
  - In this class we’ll focus on auctions (aka resource allocation), but most of the tools we’ll develop are applicable more generally

OLYMPICS 2012: A CAUTIONARY TALE

- 4 groups: A, B, C, D
- 4 teams per group
- Phase 1: Round robin within each group
  - Top two from each group advance in the second phase
- Phase 2: Knockout
  - In the first match , top team from group A is matched with second best of group C. Top team in C with second best from A. Similarly for B and D.
- What does a team want?
  - Maximize probability of winning a gold medal!
- What does the Olympic committee want?

OLYMPICS 2012: A CAUTIONARY TALE

- Phase 1:
  - What if teams A_1 and A_2 have destroyed teams A_3 and A_4, and in the final match are playing each other?
  - No problem! The loser would play the best in C, so A_1 and A_2 are still incentivized to try hard!
  - No problem? What if there's a huge upset in group C, and the (actually) best team ends up in second place?
  - Come on… What are the chances??

OLYMPICS 2012: A CAUTIONARY TALE



Video (17:30) : https://youtu.be/7mq1ioqiWEo

QUESTIONS

- When can we design systems that are robust to strategic manipulation?
- What does computer science bring to the table?
  - How much harder is mechanism design than algorithm design?
- Tradeoffs between simplicity and optimality.

ASSUMPTIONS

- We'll be working in a setting with **money**.
- Agents are **risk neutral:**
  - Value v_i with probability q_i for $i = 1, \dots, n$ is the same as value $\sum_{i=1}^{n} v_i q_i$ deterministically
- Agents have **quasi-linear** utilities:
  - Utility for value v for a price of p equals $v - p$
- We'll focus on **truthfulness**: reporting your true value maximizes your utility (more on this later)
- We'll also ask for **Individual Rationality:** if you say the truth, expected utility (over the randomness of the mechanism) is non-negative.
  - Participating is better than staying home.

AUCTIONS

We will mostly talk about auctions







AUCTIONS: EXAMPLES



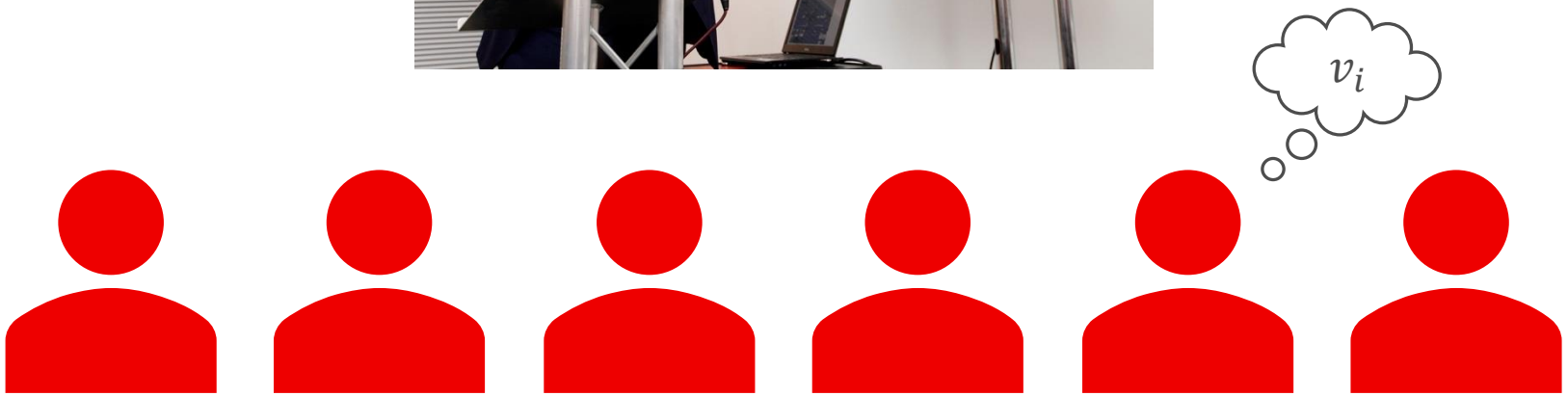




SINGLE ITEM AUCTIONS

- Single item for sale.
- n potential buyers: the bidders.
- Each bidder has a private value v_i for the item.





SEALED-BID AUCTIONS

1. Each bidder i privately communicates her bid b_i, possibly different than v_i, to the auctioneer (in a sealed envelope)
2. The auctioneer decides who to allocate the item to.
3. The auctioneer decides who pays what.

- Utility of agent $= v_i \cdot \mathbb{I}\{i \; got \; the \; item\} - p_i$

SEALED-BID AUCTIONS

- Obvious answer to (2): give the item to the highest bidder
- Reasonable ways to implement (3):
  - Highest bidder pays her bid, aka a **first price auction.**
  - Highest bidder pays the minimum bid required to win, i.e. the second highest bid. This is the **second price auction**.

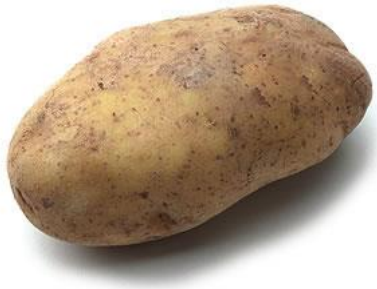
STRAWMAN

- Wait… Why charge in the first place?
- Proposal: give the item to the highest bidder and charge them nothing.
- Aka, “who can name the highest number?”

FIRST PRICE AUCTIONS

- How do I bid??
- If I bid my true value v_i I always get utility zero!
  - If I lose, I get nothing and pay nothing.
  - If I win, I pay v_i and get value v_i.
- So, I ``should'' bid something smaller than v_i
- How much smaller?

EXAMPLE



Poll 1

Assume your value = month + day of your birthday. E.g. 10/08/1997, value = 18.
How much would you bid?



FIRST PRICE AUCTIONS

- In order to argue about bidding behavior, we need to make more assumptions about the **information** agents have about other agents' bids.
- Common assumption: values come from known distribution D_i.
- Common question: what is an equilibrium bidding strategy?

SECOND PRICE AUCTIONS

- Who gets the item: highest bidder.
- What do they pay: the second highest bid.
- Claim: For a bidder to set $b_i = v_i$ (weakly) maximizes her utility *no matter what* everyone else is doing!
  - Aka being truthful is a dominant strategy!

SECOND PRICE AUCTIONS

- Claim: Truth-telling is a dominant strategy.

Proof:

- Let $b_{-i} = (\mathrm{b}_1, \ldots, \mathrm{b}_{\mathrm{i}-1}, \mathrm{b}_{i+1}, \ldots, b_n)$ be the bids of all players except i. Let $B = \max_{j \neq i} b_j$
- There are two possible outcomes:
  1. i loses and gets utility $u_i = 0$
  2. i wins, pays B and gets utility $u_i = v_i - \mathrm{B}$
- Effectively, i is picking between 0 and $v_i - B$
  - If $v_i < B, \max\{0, v_i - B\} = 0$, which you can get by bidding $b_i = v_i$
  - If $v_i \geq \mathrm{B}, \max\{0, v_i - B\} = v_i - B$, which you can get by bidding $b_i = v_i$

SECOND PRICE AUCTIONS

- Theorem: The second price auction, aka the **Vickrey auction**, is awesome!
  - Dominant strategy incentive compatible (DSIC)!
  - Maximizes Social surplus! That is, the item always goes to the agent with the highest value!
  - Can be computed in polynomial (linear) time!



TOWARDS A MORE GENERAL RESULT

- If we have a single item and want to give it to the agent with the highest value, we can do so truthfully.
- What if we don’t want to give the item to the agent with the highest value?

SINGLE PARAMETER ENVIRONMENTS

- n buyers
- Buyer i has private valuation v_i and submits a bid b_i
- An auction is a pair of two functions (x, p)
- $x(b_1, \dots, b_n) = (x_1, \dots, x_n)$ is the **allocation** function.
  - $x_i =$ Probability that item goes to player i.
  - For single item auctions: $\sum_i x_i \leq 1$
  - Our next result will not use this fact!
- $p(b_1, \dots, b_n) = (p_1, \dots, p_n)$ is the **payment** function.
  - $p_i =$ Price player i pays.

MYERSON'S LEMMA

- Definition: An allocation rule x is implementable if there is a payment rule p such that the auction (x, p) is DSIC.
- We've seen that the allocation rule ``give the item to the highest bidder'' is implementable!
- What about the allocation rule ``give the item to the 3-rd highest bidder''?

MYERSON'S LEMMA

- Definition: An allocation rule x is monotone if for every bidder i and bids b_{-i} of the other agents, the allocation $x_i(b_i, b_{-i})$ is monotone non-decreasing in b_i.
- Lemma(Myerson):
  - An allocation rule is implementable iff it is monotone
  - If x is monotone, there exists a unique (up to a constant) payment rule p that makes (x, p) DSIC, given by

$$p_i(v, b_{-i}) = v x_i(v, b_{-i}) - \int_0^v x_i(z, b_{-i}) dz$$

POLL

Poll 2

Is the allocation rule “give the item to the third highest bidder” implementable?

1. Yes
2. No



MYERSON'S LEMMA: PROOF

- IC constraint between v and v':
  - $v\, x_i(v, b_{-i}) - p_i(v, b_{-i}) \geq v x_i(v', b_{-i}) - p_i(v', b_{-i})$
  - $v' x_i(v', b_{-i}) - p_i(v', b_{-i}) \geq v' x_i(v, b_{-i}) - p_i(v, b_{-i})$
- $v(x_i(v, b_{-i}) - x_i(v', b_{-i})) \geq$

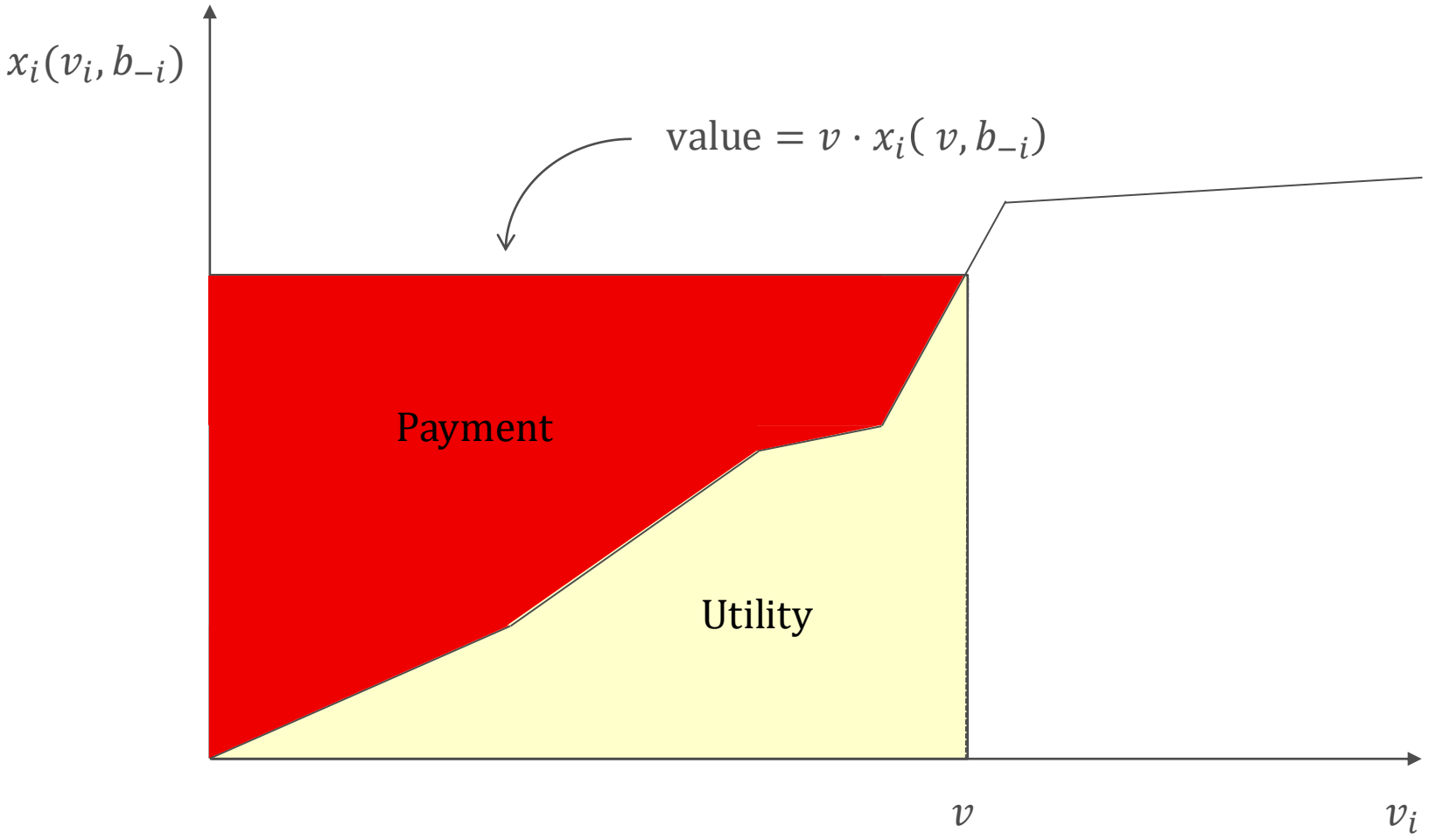
$$p_i(v, b_{-i}) - p_i(v', b_{-i})$$

$$\geq v'(x_i(v, b_{-i}) - x_i(v', b_{-i}))$$

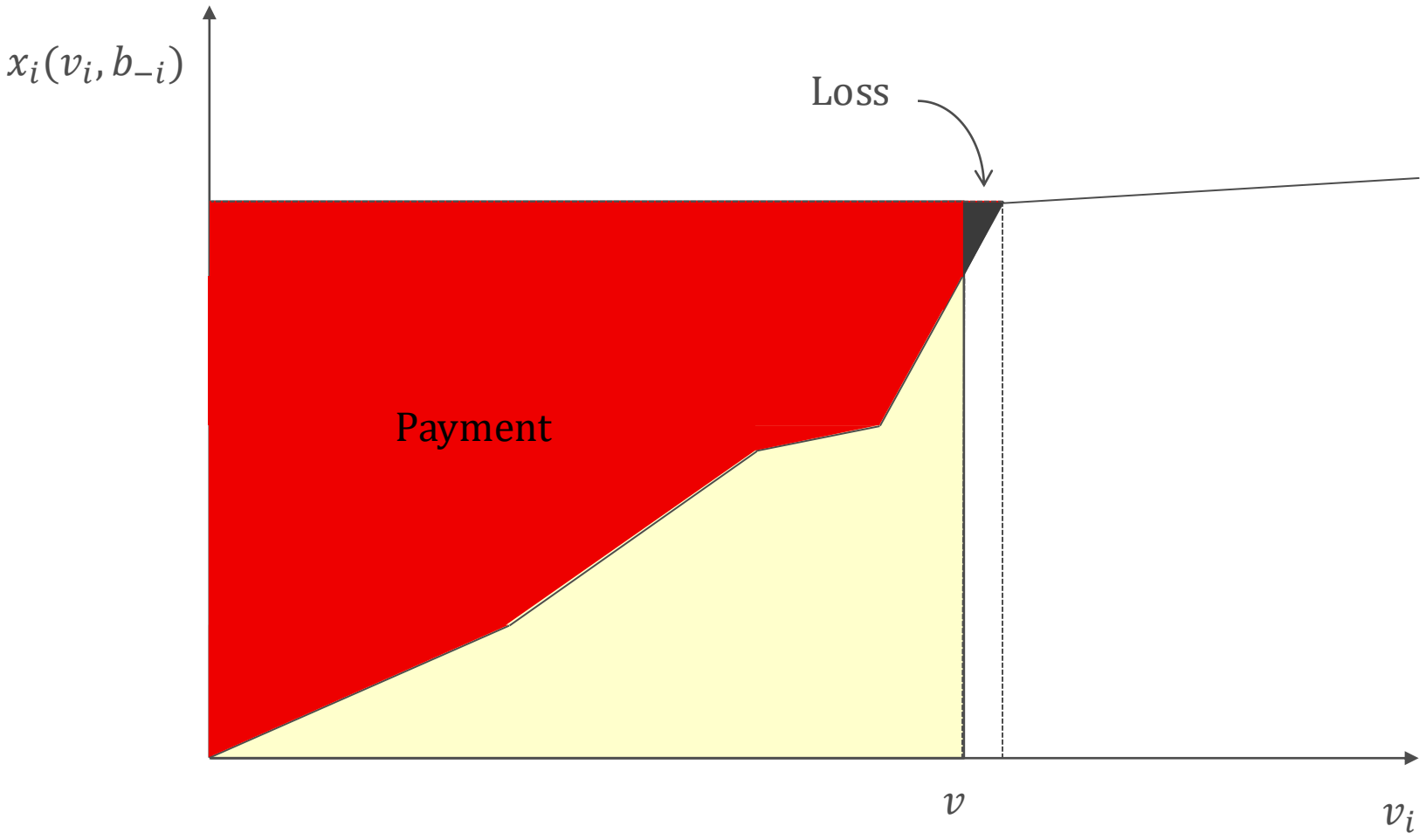
MYERSON'S LEMMA: PROOF

- $v(x_i(v, b_{-i}) - x_i(v', b_{-i})) \geq$
  $p_i(v, b_{-i}) - p_i(v', b_{-i})$
  $\geq v'(x_i(v, b_{-i}) - x_i(v', b_{-i}))$
- $v \geq v'$ implies monotonicity of the allocation!
- Take $v' = v - \epsilon$, and take the limit as ϵ goes to zero.
  - $p'_i(v, b_{-i}) = v x_i'(v, b_{-i})$
  - $p_i(v, b_{-i}) = v x_i(v, b_{-i}) - \int_0^v x_i(z, b_{-i}) dz + p_i(0, b_{-i}) + c(b_{-i})$
- Assuming that $p_i(0, b_{-i}) = 0$ (**Individual rationality**) we get the desired result.

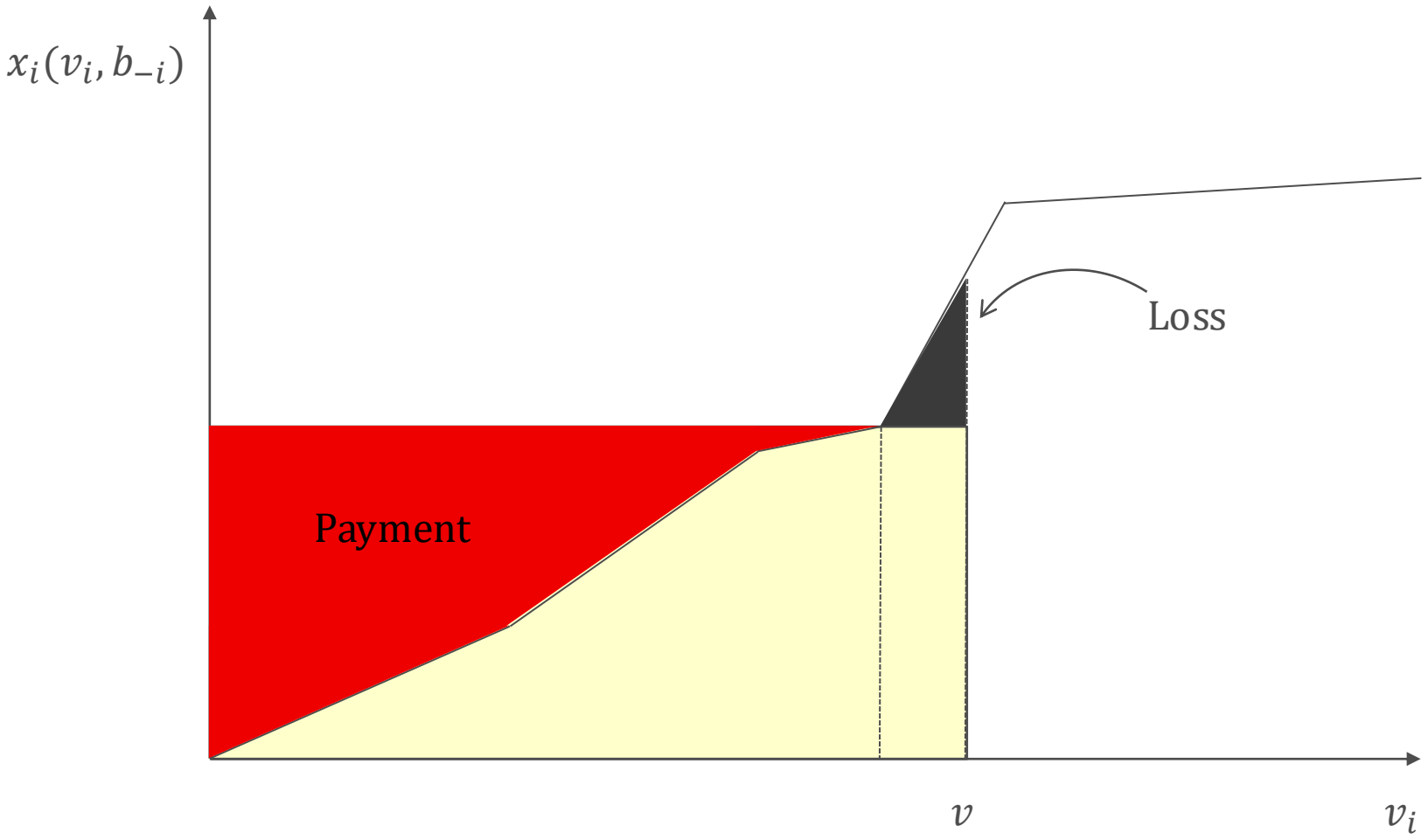
MYERSON'S LEMMA PICTORIALLY



MYERSON'S LEMMA PICTORIALLY



MYERSON'S LEMMA PICTORIALLY



SUMMARY

- Basic definitions of single parameter environments
- Second price auctions
- Myerson's lemma

REFERENCES

- Tim Roughgarden's lecture notes:
  - http://timroughgarden.org/f13/l/l2.pdf
  - http://timroughgarden.org/f13/l/l3.pd



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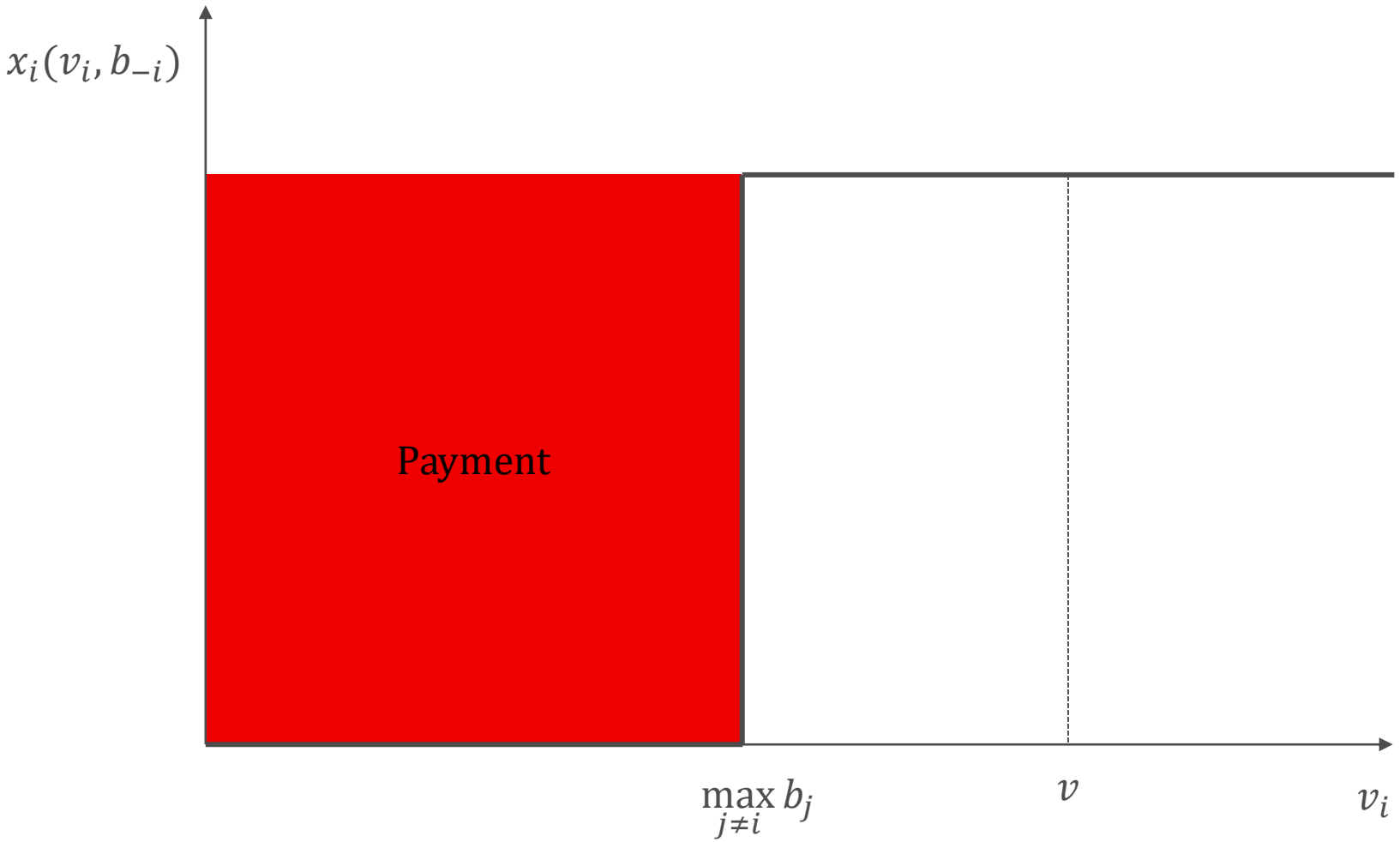
ANNOUNCEMENTS

- Homework 2 is out
  - Due Oct 6
- Midterm on Oct 8
- Project next check-in: Oct 16

LAST TIME

- Second price auctions:
  - Maximize social welfare $\sum_i v_i x_i(\vec{v})$
    - Can we give buyers more utility?
  - DSIC
  - Polytime computable
- Myerson's lemma:
  - In a single parameter environment, an allocation rule x is implementable iff it is monotone. Furthermore, there is a unique payment that makes (x, p) DSIC.

OBSERVATION: ALLOCATE TO THE BIDDER WITH THE HIGHEST VALUE



TODAY: REVENUE

- Why would we maximize social welfare?
- More reasonable to assume that sellers are trying to maximize revenue!
- For example, for $n = 1$ bidders, second price gives the item for free!
  - Pretty unreasonable…

ROGER MYERSON



MAXIMIZE REVENUE

- Focus on a single bidder, with private value v
- Make a take-it-or-leave-it offer
  - For a single bidder this is the only deterministic DSIC mechanism
- How much should we price the item at?
- If we magically knew v, we would set a price of v, but v is private…

EXAMPLE

- A price of r yields revenue r if $v \geq r$, and 0 otherwise
- A price of 10,000$ is
  - Good if v is slightly higher than 10,000$
  - Bad if v is a lot higher than 10,000$
  - Horrible if v is 9,999$

REVENUE

- Different auctions perform different on different inputs.
  - Contrast this with social welfare.
- We take a Bayesian approach!
- The private value v_i of bidder i is drawn from a **known** distribution D_i.
  - Today: distributions' support is $[0, v_{max}]$
- Goal: Maximize **expected** revenue over all DSIC and IR mechanisms.

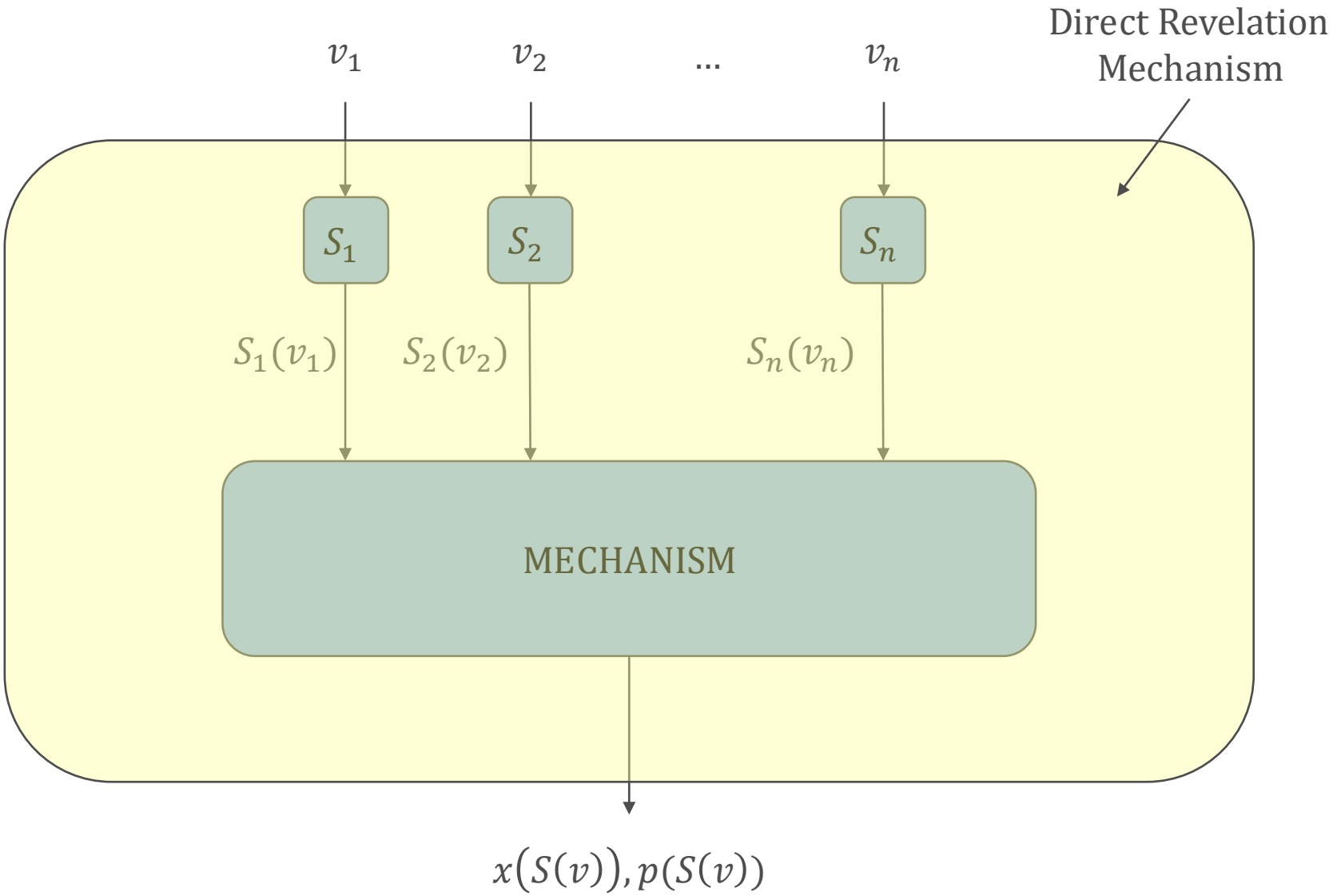
WHY DSIC?

- Easy for participants to figure out what to bid
- The seller can predict what the bidders will do assuming only that they bid their dominant strategy
  - Pretty weak behavioral assumption
- Can you make more money with a non-DSIC mechanism??
  - Today: no!
  - Generally: yes!

REVELATION PRINCIPLE

- Optimize over the space of all DSIC mechanisms???
- That sounds super hard…
- We will focus on **direct revelation mechanisms**!
  - You reveal your private information to the system.
  - As opposed to setting up a weird auction, where agents have dominant strategies
- This happens to be without loss of generality for a weaker IC notion
  - It turns out that the optimal mechanism under the weaker notion is also DSIC

REVELATION PRINCIPLE



THE GAME

1. Seller is told distributions D_i for each buyer
2. Seller commits to a DSIC auction (x, p)
3. Nature draws v_i from D_i.
   - Today: independent D_is
4. Each agent i learns v_i
5. Each agent i submits bid b_i
6. Item is allocated according to $x(\vec{b})$, and payments are transferred according to $p(\vec{b})$

Goal:

- We take the seller's perspective.
- Design a DSIC and IR auction that maximizes **expected** revenue (expectation with respect to randomness in D and randomness in the auction)

SINGLE BUYER

- Expected revenue from setting a price of p

$$p \cdot \Pr[v \geq p] = p \cdot (1 - F(p))$$

- Say $D = U[0,1]$
- $Rev(p) = p \cdot \big(1 - F(p)\big) = p \cdot (1 - p)$
- $Rev'(p) = -2p + 1 = 0 \rightarrow p = \frac{1}{2}$
- Expected revenue $= \frac{1}{4}$
- This is optimal!
- What about two bidders??

TWO BIDDERS

- Say $D_1 = D_2 = D = U[0,1]$
- We could run a second price auction…
- What's the expected revenue?
- Observation: $E[Rev] = E[\min\{v_1, v_2\}]$
- $\Pr[\min\{v_1, v_2\} \geq x] = \Pr[v_1 \geq x \;\&\; v_2 \geq x]$

  $= \Pr[v_1 \geq x] \cdot \Pr[v_2 \geq \mathrm{x}]$

  $= (1 - x)^2$
- $E[min] = \int_{x=0}^{1} \Pr[min \geq x]\, dx = 1/3$

TWO BIDDERS

- $D_1 = D_2 = D = U[0,1]$
- Second price auction gives $1/3$
- Can we do better?
- What if we never sell under ½?
  - Similar to what we did for one buyer.
- If highest bid $> \frac{1}{2}$: Highest bidder pays the maximum of ½ and the second highest bid
- If highest bid $< \frac{1}{2}$: No one gets the item
- Expected revenue of this auction is $\frac{5}{12} > \frac{1}{3}$
- Can we do better???

MYERSON

- The expected revenue of a DSIC auction (x, p) is equal to

$$\mathbb{E}_{\vec{v} \sim D}\left[\sum_{i=1}^{n} p_i(\vec{v})\right]$$

- For this results we assume **independent** buyer distributions.
- **Goal**: give a formula for the expected revenue that's easier to maximize!

MYERSON

- Step 0: Move things around:

$$\mathbb{E}_{\vec{v}\sim D}\left[\sum_{i=1}^{n} p_i(\vec{v})\right] = \sum_{i=1}^{n} \mathbb{E}_{v_{-i}}\left[\mathbb{E}_{v_i}\left[p_i(v_i, v_{-i})\right]\right]$$

MYERSON

- $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} p_i(\vec{v})] = \sum_{i=1}^{n} \mathbb{E}_{v_{-i}}[\mathbb{E}_{v_i}[p_i(v_i, v_{-i})]]$
- Step 1: Apply Myerson's lemma

$$p_i(v, b_{-i}) = v x_i(v, b_{-i}) - \int_0^v x_i(z, b_{-i}) dz$$

- $\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] = \int_0^{v_{\max}} p_i(v_i, v_{-i}) f_i(v_i) dv_i$

$$= \int_0^{v_{\max}} \left(v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_{-i}) dz \right) f_i(v_i) dv_i$$

$$= \int_0^{v_{max}} v_i \, x_i(v_i, v_{-i}) f_i(v_i) dv_i -$$

$$\int_0^{v_{max}} \int_0^{v_i} x_i(z, v_{-i}) \, f_i(v_i) dz dv_i$$

MYERSON

$$\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] = \int_0^{v_{max}} v_i \, x_i(v_i, v_{-i}) f_i(v_i) dv_i - \int_0^{v_{max}} \int_0^{v_i} x_i(z, v_{-i}) \, f_i(v_i) dz dv_i$$

- Step 2: Change order of integration
- $\int_0^{v_{max}} \int_0^{v_i} x_i(z, v_{-i}) \, f_i(v_i) dz dv_i$

$$= \int_0^{v_{max}} x_i(z, v_{-i}) \int_z^{v_{max}} f_i(v_i) dv_i dz$$

$$= \int_0^{v_{max}} x_i(z, v_{-i})(1 - F_i(z)) dz$$

MYERSON

$$\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] = \int_0^{v_{max}} v_i \, x_i(v_i, v_{-i}) f_i(v_i) dv_i - \int_0^{v_{max}} x_i(v_i, v_{-i})(1 - F_i(v_i)) dv_i$$

- Step 3: Combine
- $\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] =$

$$\int_0^{v_{max}} f_i(v_i) x_i(v_i, v_{-i}) \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) dv_i$$

MYERSON

- $\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] =$

$$\int_0^{v_{max}} f_i(v_i) x_i(v_i, v_{-i}) \left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right) dv_i$$

- Step 4: A definition:

The **virtual value** of bidder i is

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

MYERSON

$$\mathbb{E}_{v_i}[p_i(v_i, v_{-i})] = \mathbb{E}_{v_i}[x_i(v_i, v_{-i}) \cdot \phi_i(v_i)]$$

$$\text{where } \phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$$

- Step 5: Plug everything back:

$$\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} p_i(\vec{v})] = \sum_{i=1}^{n} \mathbb{E}_{v_{-i}}[\mathbb{E}_{v_i}[p_i(v_i, v_{-i})]]$$

$$= \mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})]$$

MYERSON

- $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} p_i(\vec{v})] =$

$$\mathbb{E}_{\vec{v}\sim D}\left[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})\right]$$

- Ok, let's parse this…
- Maximizing expected revenue is the same as maximizing the expected virtual welfare!
- We (kind of) already know how to solve that!
- **Second price auction** (but in virtual value space).

MYERSON

- Old problem:

$$\max \mathbb{E}_{\vec{v}\sim D}\left[\sum_{i=1}^{n} p_i(\vec{v})\right]$$

Subject to

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i x_i(v', v_{-i}) - p_i(v', v_{-i})$$

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0$$

$$\sum_i x_i(\vec{v}) \leq 1$$

- New problem:

$$\mathbb{E}_{\vec{v}\sim D}\left[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})\right]$$

Subject to

$$\sum_i x_i(\vec{v}) \leq 1$$

$x_i(.)$ is a monotone function

MYERSON

- Maximize $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})]$
- We can maximize this pointwise!

MYERSON

- Example: $n = 2$, D_1 and D_2 have support size $\{0,1\}$
- Maximize

$$\Pr[v_1 = 0, v_2 = 0](\phi_1(0)x_1(0,0) + \phi_2(0)x_2(0,0)) +$$
$$\Pr[v_1 = 0, v_2 = 1](\phi_1(0)x_1(0,1) + \phi_2(1)x_2(0,1)) +$$
$$\Pr[v_1 = 1, v_2 = 0](\phi_1(1)x_1(1,0) + \phi_2(0)x_2(1,0)) +$$
$$\Pr[v_1 = 1, v_2 = 1](\phi_1(1)x_1(1,1) + \phi_2(1)x_2(1,1))$$

- Subject to

$$x_1(i,j) + x_2(i,j) \leq 1, for\ all\ i,j \in \{0,1\}$$

MYERSON

- Maximize $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})]$
- We can maximize this pointwise!
- Who gets the item?
  - **Highest virtual value!**
- How much do they pay?
  - Second ~~highest~~ virtual value??
  - **The value they would have to bid in order to lose!**

Kind of…

POLL

- Maximize $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})]$
- $\phi_1(v_1) = v_1 - 1$
- $\phi_2(v_2) = v_2 - 1$
- $v_1 = 1/2$
- $v_2 = 1/4$

Poll 1

Who gets the item?

1. 1
2. 2
3. Half, half
4. Neither



POLL

- Maximize $\mathbb{E}_{\vec{v}\sim D}[\sum_{i=1}^{n} \phi_i(v_i) \cdot x_i(v_i, v_{-i})]$
- $\phi_1(v_1) = 2v_1 - 1$
- $\phi_2(v_2) = v_2 - 1$
- $v_1 = 1$
- $v_2 = 1/4$

Poll 2

Who gets the item? How much do they pay?



1. 1, -3/4
2. 1, 1/2
3. 1, 0
4. 1, 1/4

MYERSON

- Allocate to the agent with the highest virtual value (if it's non-negative).
- **No!** The allocation rule might not be monotone!
  - $\phi_i(v)$ might decrease as v increases
- Myerson provided a solution to this: "iron" the virtual value function.
  - We won't cover this.

MYERSON

- *Definition*: A distribution with cdf F and pdf f is called **regular** if $\phi(v) = v - \frac{1-F(v)}{f(v)}$ is monotone non-decreasing
  - If $\frac{1-F(v)}{f(v)}$ is monotone non-increasing we say that the distribution has **monotone hazard rate** (MHR).
- Most distributions you know are regular (and MHR): uniform, exponential, Normal, Gamma, etc etc.
- Intuitively, regular = small tail

MYERSON: REGULAR DISTRIBUTIONS

- Give the item to the agent with the highest virtual value, or no one if all virtual values are negative.
  - Good news: monotone allocation rule
  - Weird news: Highest virtual value ≠ highest value!
- $\phi_1(v_1) = 2v_1 - 1, \phi_2(v_2) = 2v_2 - 100$
  - $v_1 = 0.6, v_2 = 50 \rightarrow$ Agent 1 wins!

MYERSON: REGULAR DISTRIBUTIONS

- Give the item to the agent with the highest virtual value, or no one if all virtual values are negative.
- If the item was given to agent i
  - Let j be the agent with the second highest virtual value
  - If $\phi_j(v_j) < 0$, i pays $\phi_i^{-1}(0)$
  - If $\phi_j(v_j) \geq 0$, i pays $\phi_i^{-1}\left(\phi_j(v_j)\right)$
- Different way to think about it:
  - Seller inserts her own bids (in v.v. space) $\phi_1^{-1}(0), \phi_2^{-1}(0), \ldots$

MYERSON: IDENTICAL REGULAR DISTRIBUTIONS

- Actually simple if all agents have the same distribution $D = D_i, \forall i$
- Give the item to the agent with the highest virtual value, or no one if all virtual values are negative.
  - Highest virtual value = Highest value
  - Rephrase: Give the item to the agent with the highest value, if her virtual value is non-negative.
- If the item was given to agent i, she pays the maximum of the second highest bid and $\phi^{-1}(0)$
- In other words, the optimal auction is a second price auction with a reserve of $\phi^{-1}(0)$
  - Does this look familiar?
  - Precisely the E-Bay format!

EXAMPLE

- 2 agents, $D_1 = D_2 = D = U[0,1]$
- $\phi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v - 1$
- Allocation rule: give it to the person with the highest virtual value $\phi(v_i)$, if its non-negative
- Aka, give it to the person with the highest value v_i, if its at least ½
- Charge $\max\{½, other\ bid\}$

SUMMARY

- Single parameter environments
  - Second price auctions
  - Myerson's lemma
  - Myerson's optimal auction

REFERENCES

- Tim Roughgarden's lecture notes:
  - http://timroughgarden.org/f13/l/l4.pdf
  - http://timroughgarden.org/f13/l/l5.pdf



Algorithmic Game Theory

CS 58100 · Mechanism Design III

Alex Psomas

Fall 2026

TODAY

- Bulow-Klemperer
- Cremer-McLean for correlated buyers
- Prophet Inequalities

CRITIQUE #1: TOO MUCH DEPENDENCE ON THE DISTRIBUTION

- Optimal auction depends on the distribution
- Wasn't the whole point of the Bayesian approach that this is unavoidable?
- We'll assume that $v_i \sim D_i$ (in the analysis), but our auctions will **not** depend on the $D_i s$
  - "Prior independent" mechanism design

PRIOR INDEPENDENT MECHANISMS

- Sounds pretty optimistic…
- Existence of a good prior independent auction *A* for (say) regular distributions implies that a **single** auction can compete with all the (uncountably many) optimal auctions, *tailored* to each distribution, **simultaneously**!
- Pretty wild!
- Any candidates?
  - Second price auction!

BULOW-KLEMPERER THEOREM

- $OPT(n, D)$= Expected revenue of optimal auction with n i.i.d. buyers from D.
- $V(n, D)$ = Expected revenue of Vickrey with n i.i.d. buyers from D.
- Theorem (1996): For all regular D we have

$$V(n+1, D) \geq OPT(n, D)$$

- In more modern language: "The competition complexity of single-item auctions with regular distributions is 1"
  - The competition complexity of n bidders with additive valuations over m independent, regular items is at least $log m$ and at most n + $2m - 2$ [EFFTW, EC 17]
  - At most $n(\ln\left(1+\frac{m}{n}\right)+2)$ and also at most $9\sqrt{nm}$ [BW, STOC 2019]
  - For $m < n$, the competition complexity of n bidders with additive values is exactly $\Theta(\sqrt{nm})$ [DRWX, EC 2024]

BULOW-KLEMPERER THEOREM

- Theorem (1996): For all regular D we have

$$V(n+1, D) \geq OPT(n, D)$$

- Intuitively: It is better to increase competition by a single buyer than invest in learning the underlying distribution!

BULOW-KLEMPERER THEOREM

Proof:

- Let A be the following auction for $n+1$ buyers from D:
  - Run $OPT(n, D)$ on buyers $1, \dots, n$
  - If the item is not sold, give it for free to buyer $n+1$
- Obvious observation 1: $Rev(A) = OPT(n, D)$
- Obvious observation 2: A always allocates the item.

BULOW-KLEMPERER THEOREM

- Non obvious:
- The second price auction is the revenue maximizing auction over all auctions that always allocate the item.
  - Why?
- Therefore

$$V(n+1, D) \geq Rev(A) = OPT(n, D)$$

CRITIQUE #2: MYERSON DOESN'T WORK FOR CORRELATED BUYERS

- Myerson: Optimal auction for independent bidders.
- What if the bidders' values are correlated?
  - Very realistic!
- We'll see a 2 agent instance of a result of Cremer and McLean [1998]
  - They show how to extract the full social welfare under very mild conditions on the correlation

CREMER-MCLEAN

v_1/v_2	1	2	3
1	1/6	1/12	1/12
2	1/12	1/6	1/12
3	1/12	1/12	1/6

Poll 1

How much revenue does a second price auction make (in expectation)?

1. 8/6
2. 10/6
3. 12/6
4. 14/6



CREMER-MCLEAN

v_1/v_2	1	2	3
1	1/6	1/12	1/12
2	1/12	1/6	1/12
3	1/12	1/12	1/6

Poll 2

What's the maximum possible revenue an auction can make?

1. 8/6
2. 10/6
3. 12/6
4. 14/6



CREMER-MCLEAN

v_1/v_2	1	2	3
1	1/6	1/12	1/12
2	1/12	1/6	1/12
3	1/12	1/12	1/6

- $P_{i,j} = \Pr[v_2 = j \mid v_1 = i]$

$\mathrm{P} =$

1/2	1/4	1/4
1/4	1/2	1/4
1/4	1/4	1/2

- $E[utility\ of\ v_1 = 1\ from\ SP] = 0$
- $E[utility\ of\ v_1 = 2\ From\ SP] = 1/4 \cdot 1 = 1/4$
- $E[utility\ of\ v_1 = 3\ from\ SP] = 1/4 \cdot 2 + 1/4 \cdot 1 = 3/4$

CREMER-MCLEAN

- Observation: P has full rank
- Therefore, $P \cdot (x_1, x_2, x_3)^T = (0, {}^1/_4, {}^3/_4)^T$ has a solution:
  - $x_1 = -1, x_2 = 0, x_3 = 2$

The magic part

- Consider the following bet B_1 for player 1:
  - I pay you 1 if $v_2 = 1$
  - Nothing happens if $v_2 = 2$
  - You pay me 2 if $v_2 = 3$

CREMER-MCLEAN

- Consider the following bet B_1 for player 1: (a) I pay you 1 if $v_2 = 1$, (b) Nothing happens if $v_2 = 2$, (c) You pay me 2 if $v_2 = 3$
- What's the expected value for taking this bet if $v_1 = 1$?
  - $1/2 \cdot 1 + 1/4 \cdot 0 + 1/4 \cdot (-2) = 0$
- What if $v_1 = 2$? $-1/4$
- What if $v_1 = 3$? $-3/4$
- Similar bet B_2 for player 2
- Auction: Player i is offered bet B_i. After the bet we'll run a second price auction
  - $E[utility\ of\ v_1 = 1] = E[utility\ of\ B_1] + E[utility\ from\ SP] = 0$
  - $E[ut.\ of\ v_1 = 2] = -1/4 + 1/4 = 0$
  - $E[ut.\ of\ v_1 = 3] = -3/4 + 3/4 = 0$

CREMER-MCLEAN

- Since buyers always have zero utility, and the item is always sold, the seller must be extracting all of the social welfare
- Expected revenue $= 14/6$
- Wth just happened???
- That's a pretty weird auction!
- This "prediction" is very unlikely to be observed in practice.
- Research direction (project idea?): find a way out

CRITIQUE #3 FOR MYERSON: OPTIMAL AUCTIONS ARE WEIRD

- $n = 2. D_1 = U[0,1], \mathrm{D}_2 = \mathrm{U}[0,100]$
- $\phi_1(v_1) = 2v_1 - 1, \phi_2(v_2) = 2v_2 - 100$
- Optimal auction
  - When $v_1 \leq 1/2$ and $v_2 \geq 50$: Sell to 2 for 50
  - When $v_1 > 1/2$ and $v_2 < 50$: Sell to 1 for ½
  - When $0 < 2v_1 - 1 < 2v_2 - 100$: Sell to 2 for $(99+2v_1)/2$ (slightly over 50)
  - When $0 < 2v_2 - 100 < 2v_1 - 1$: Sell to 1 for $(2v_2 - 99)/2$ (slightly over ½)
- Wth is this???
- Impossible to explain, unless you go through all of Myerson's calculations!

CRITIQUE #3: OPTIMAL AUCTIONS ARE WEIRD

- Weirdness inevitable if you want optimality
- Weirdness inevitable if you're 100% confident in the model
- Take away: Optimality requires complexity
- In the remainder: ask for simplicity and settle for approximately optimal auctions.

CRITIQUE #3: OPTIMAL AUCTIONS ARE WEIRD

A (cool) detour: Prophet inequalities!

PROPHET INEQUALITY

- n treasure boxes.
- Treasure in box i is distributed according to known distribution D_i
- In stage i you open box i and see the treasure (realization of the random variable) x_i
- After seeing x_i you either take it, or discard it forever and move on to stage $i + 1$
- What should you do?
- Our goal will be to compete against a prophet who knows the realizations of the D_is

PROPHET INEQUALITY






$D_1 = U[0,60]$ $D_2 = Exp[1/60]$ $D_3 = N[1,1]$ $D_4 = U[0,100]$



$x_1 = 54$ $x_2 = 52$ $x_3 = 1$ $x_4 = 61$



$x_1 = 54$

$x_2 = 52$ 

Our value is 52, Prophet gets 61

PROPHET INEQUALITY

- Optimal policy: Solve it backwards!
  - If we get to the last box, we should clearly take x_n
  - For the second to last, we should take x_{n-1} if it's larger than $E[x_n]$
  - We should take x_{n-2} only if it's larger than the expected value of the optimal policy starting at $n-1$, i.e. $\Pr[x_{n-1} > E[x_n]] \cdot E[x_{n-1} | x_{n-1} > E[x_n]] + \Pr[x_{n-1} \leq E[x_n]] \cdot E[x_n]$
  - And so on…
- Ok, that's pretty complicated…
- Any simpler policies?
  - Focus on policies that set a single threshold t and accept x_i if it's above t, otherwise reject
  - How good are those?

RESOURCES

- Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions. Jacques Crémer and Richard P. McLean. Econometrica 1988
- Tim Roughgarden's notes:
  - http://timroughgarden.org/f13/l/l6.pdf
- (Perhaps helpful) Gil Kalai's and Jason Hartline's response in this thread:
  - https://math.stackexchange.com/questions/139575/can-someone-explain-cremer-mcleans-astonishing-result-in-auction-theory



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$x_1 = 54$

$x_2 = 52$ 

Our value is 52, Prophet gets 61

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  - And so on…
- Ok, that's pretty complicated…
- Any simpler policies?
  - Focus on policies that set a single threshold t and accept x_i if it's above t, otherwise reject
  - How good are those?

PROPHET INEQUALITY

- **Theorem**: There exists a single threshold t^* such that the policy that accepts x_i when $x_i \geq t^*$ gives expected reward at least $\frac{1}{2} E[\max_i x_i]$, i.e. at least half of what the prophet makes (in expectation).

PROPHET INEQUALITY

Proof

- $Z^+ = \max\{z, 0\}$
- Given a "threshold policy" with threshold t, let $q(t) = \Pr[policy\ accepts\ no\ prize]$
- Large t: large $q(t)$, but big rewards
- Small t: small $q(t)$, but small rewards
- $E[reward] \geq q(t) \cdot 0 + \big(1 - q(t)\big) \cdot t$
- A little too pessimistic…
- When $x_i \geq t$ we'll count x_i, not t

PROPHET INEQUALITY

$$E[reward] = t\big(1 - q(t)\big) +$$

$$\sum_i E[x_i - t | x_i \geq t \;\&\; x_j < t, \forall j \neq i] \cdot \Pr[x_i \geq t \;\&\; x_j < t, \forall j \neq i]$$

$$= t\big(1 - q(t)\big) +$$

$$\sum_i E[x_i - t | x_i \geq t] \cdot \Pr[x_i \geq t] \cdot \Pr\big[\, x_j < t, \forall j \neq i\big]$$

$$= t\big(1 - q(t)\big) + \sum_i E[(x_i - t)^+] \cdot \Pr[x_j < t, \forall j \neq i]$$

$$\geq t\big(1 - q(t)\big) + q(t) \sum_i E[(x_i - t)^+]$$

(we used that $q(t) = \Pr\big[x_j < t\,, \forall j\big] \leq \Pr[x_j < t, \forall j \neq \mathrm{i}]$)

PROPHET INEQUALITY

$$E[reward] \geq t\big(1 - q(t)\big) + q(t) \sum_i E[(x_i - t)^+]$$

$$\begin{aligned} E[\max_i x_i] &= E[t + \max_i (x_i - t)] \\ &= t + \ E[\max_i (x_i - t)] \\ &\leq t + E[\max_i (x_i - t)^+] \\ &\leq t + \textstyle\sum_i E[(x_i - t)^+] \end{aligned}$$

$$t^*: q(t^*) = {}^1\!/_2$$

$$E[reward] \geq \frac{t^*}{2} + \frac{1}{2}\sum_i E[(x_i - t^*)^+] \geq \frac{1}{2} E[\max_i x_i]$$

BACK TO AUCTIONS

- $Rev = E[\sum_i \phi_i(v_i)x_i(v_i)] = E[\max_i \phi_i(v_i)^+]$
  - Kind of like the reward of the prophet
- Pick t^* such that $\Pr[\max_i \phi_i(v_i)^+ \geq t^*] = 1/2$
- Give item to bidder i if $\phi_i(v_i) \geq t^*$
- Prophet inequality gives

$$E[reward] = E[\sum_i \phi_i(v_i)x_i(v_i)] \geq \frac{1}{2} E[\max_i \phi_i(v_i)^+]$$

- More concretely:
  - $r_i = \phi_i^{-1}(t^*)$
  - Remove all bidders with $b_i < r_i$
  - Run a second price with the remaining bidders

MODERN PROPHETS

- The proof we saw is due to Ester Samuel-Cahn [1984]
- These days people are interested in prophet inequalities for more complex settings
- E.g.:
  - There are m items
  - n buyers arriving over time, one per stage; we know distributions, but learn values at arrival
  - When buyer i arrives we put prices on the items (possibly tailored for i), and then i buys their favorite bundle given the prices
  - Question: Can we get a constant approximation to the prophet?
  - Yes!
    - For submodular and XOS valuations this was known since 2017 or so. See [Dutting et al. FOCS 2017]
    - For subadditive valuations, see [Corea, Cristi; STOC 2023]. This result is non-constructive! Algorithmic result open.

MORE MODERN PROOF OF PROPHET

- Set a price of p
- Give it to the first agent/box that has value at least p
- Welfare = Revenue + Surplus
  - Surplus = "buyer happiness", value minus price
- Revenue $= p \cdot \Pr[item\ is\ sold]$
- Surplus $\geq \sum_i \Pr[i\ has\ a\ chance\ to\ buy]\ E[(v_i - p)^+]$
  - $\Pr[i\ has\ a\ chance\ to\ buy] \geq \Pr[\ item\ is\ not\ sold]$
  - $\sum_i E[(v_i - p)^+] \geq E[\max_i (v_i - p)^+] \geq E[\max_i v_i] - p$
- Overall, welfare $\geq$

$$p \cdot \Pr[item\ is\ sold] + \Pr[\ item\ is\ not\ sold] \cdot (E[\max_i v_i] - p)$$

- Set $p = E\left[\max_i v_i\right]/2$ and you're done!

SO FAR

- Revelation Principle
- Single parameter environments
  - Second price auctions
  - Myerson's lemma
  - Myerson's optimal auction
  - Cremer-McLean auction for correlated buyers
  - Prophet inequalities
  - Bulow-Klemperer

TODAY

- Multidimensional environments
  - The VCG mechanism
  - Challenges
  - Revenue optimal auctions are strange

COMBINATORIAL AUCTIONS

- n buyers
- m items
  - Result holds for arbitrary feasibility sets
- Agent i has an (arbitrary) value function v_i from subsets of items to non-negative reals
  - Complementarities: $v_i(\{1\}) = 1, v_i(\{2\}) = 2, v_i(\{1,2$ 100
  - Substitutes: $v_i(\{1\}) = 10, v_i(\{2\}) = 10, v_i(\{1,2\}) =$ 12
  - Or anything else really: $v_i(\{1\}) = 10, v_i(\{2\}) = 10, v$ 0

MAXIMIZE SOCIAL WELFARE

- Goal: find a truthful (DSIC) and feasible mechanism that maximizes social welfare $max_{feasible\ S=(S_1,\ldots,S_n)} \sum_i v_i(S_i)$

MAXIMIZE SOCIAL WELFARE

- Let's try our single dimensional approach
- Commit to the allocation rule x that maximizes social welfare, and then find appropriate payments
- $x(\vec{b}) = argmax_S \sum_i b_i(S_i)$
  - b_i here is a reported *valuation function,* not a number
- Let S^* be the social welfare maximizing allocation
- What about payments?
- Perhaps try to prove another "Myerson's Lemma"?
- Not clear what monotonicity is...
- Not clear what the "critical bid" is (bids are functions)…

VICKREY-CLARKE-GROVES

- Key idea: "**externality**"
- How much pain does your existence cause?
- Aka, how much does your presence hurt everyone else's value?
- That's how much you should pay
- In retrospect, same idea as second price auction
  - I pay exactly the maximum social welfare when I'm not there

VICKREY-CLARKE-GROVES

$$p_i(\vec{b}) = \underbrace{\max_S \sum_{j \neq i} b_j(S_j)}_{\text{Without } i} - \underbrace{\sum_{j \neq i} b_j(S_j^*)}_{\text{With } i}$$

- $p_i(\vec{b}) \geq 0$
  - Why?
- Claim: VCG is DSIC

VICKREY-CLARKE-GROVES

Proof:

- Fix i and b_{-i}
- Let S^* be the social welfare maximizing allocation when i submits b_i

$$u_i = \underbrace{v_i(S_i^*) + \sum_{j \neq \mathrm{i}} b_j(S_j^*)}_{\text{(A)}} \underbrace{- \max_S \sum_{j \neq i} b_j(S_j)}_{\text{(B)}}$$

- Observation: (B) is out of i's control
- Thought experiment: i can directly pick any S^*
  - As opposed to picking b_i (that affects S^*)
- What would she pick?
  - Maximize (A)
- What is our mechanism doing?