Introduction to Discrete Fair Division

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Introduction to Discrete Fair Division 1

Motivations

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- People want stuff
- But there is only a finite amount of things

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Motivations

- People want stuff
- But there is only a finite amount of things
- This is the fundamental question of economics
  - How do we study the movement of resources within a society?

The Computational Problem

- It is a computationally difficult problem to distribute items among people

- How can we guarantee truthfulness in our algorithm? (People may lie about their value of a good)

Applications

Variations of this problem have been used for

1. Food distribution
   - Indy Hunger Network
   - Foodbank Australia
2. Computing resources
   - Memory
   - CPU Time
3. Estate Divisions
   - Spliddit

Premise

Premise

- How can we allocate a set of goods to a set of people fairly?
- This is surprisingly complicated
  - People may like different goods differently
  - What does fairness even mean?

Discrete Setting

- We define N (of cardinality n) to be the set of agents and M (of cardinality m) to be the set of goods.

- Each $i \in N$ is equipped with a valuation function v_i, which assigns a positive valuation to each subset of M

$$v_i : 2^M \rightarrow \mathbb{R}_{\geq 0}$$

- Intuitively, valuation $v_i(\varnothing) = 0$
- An allocation is a partition of M

$$X =< X_1, X_2, ...X_n >$$

- v_i is additive $v_i(A \cup B) = v_i(A) + v_i(B)$

EF and PROP

Notions of Fairness

- Proportionality
  - Each agent believes they receive at least $\frac{1}{n}$ of the goods

$v_i(X_i) \geq \frac{1}{n} \times v_i(M)$

- Envy-Freeness (**EF**)
  - Each agent believes they receive weakly more than the other agents

$$\forall_{i,j \in N} v_i(X_i) \geq v_i\left(X_j\right)$$

Notions of Fairness

Example: In a discrete setting, EF allocations may not always exist

Proof: By counterexample, take 1 good and 2 agents. We arbitrarily give agent 1 the good

$$v_2(X_1) > 0 = v_2(X_2)$$

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A similar argument can be applied to proportionality

Lemma 0.1: EF $\rightarrow$ PROP for additive valuations

Proof: Assume that there is an envy free allocation and non proportional allocation. Every player believes that they have a piece of equal or better value to that of everyone else. Because the $v_i(X_j)$ may have value < 1/n, and there are n player, this means that $v_i(M)$ can be $< v_i(M)$ ■

Relaxing EF

- Envy Free up to X (EFX)
  - Each agent believes they receive weakly more than the other agents without some good

$$\forall_{i,j} \forall_{g \in X_j} v_i(X_i) \geq v_i\big(X_j \setminus \{g\}\big) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i\big(X_j \setminus \min\big(X_j\big)\big)$$

- It is not clear if EFX allocations exist or can be computed in polynomial time in general
  - Several relaxations of EFX have been proposed

Known cases for EFX

- EFX can be computed efficiently for n=2
  - Cut and Choose
- An algorithm exists to compute an EFX allocation in pseudo polynomial time for n=3

Relaxations

Relaxing EFX

- Realized valuations
  - Shared valuation function v
  - EFX exists and a polynomial time algorithm is known
- EF1

$$\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq v_i\big(X_j \setminus \{g\}\big) \leftrightarrow \forall_{i,j} v_i(X_i) \geq v_i\big(X_j \setminus \max\big(X_j\big)\big)$$

- α-EFX
  - $$\forall_{i,j} \exists_{g \in X_j} v_i(X_i) \geq \alpha \times v_i\big(X_j \setminus \{g\}\big)$$

- If EFX allocations exist in general, then $\alpha = 1$

Break Time!

Let's Cut a Cake

Other Fairness

Maximin Fair Share

- Let $X_n(M)$ be the set of possible allocations of M goods to n agents
- $\mu_i^n(M)$ is one of the partitions which maximizes the least valuable bundle according to i

$$\mu_i^n(M) = \max_{B \in X_n(M)} \min_{S \in B} v_i(S)$$

- Maximin Share Fair (MMS) allocations do not always exist

Relaxing MMS

- α-MMS $v_i(X_i)$

Pareto Optimality

- An allocation is Pareto Optimal (PO) iff an agent would protest to another allocation
- The allocation A is PO if there is no allocation B such that $v_i(B_i) \geq v_i(A_i)$ for all i and one inequality is strict
- A is not *Pareto Dominated* by another allocation

MNW

- Maximize the geometric mean of values for agents with positive value

$$\left(\prod_{i}^{n} v_i(X_i) \right)^{\frac{1}{n}}$$

- MNW $\rightarrow$ EF1 $\wedge$ PO

  Proof:

  ■

- Can an EF1 and PO allocation be computed in polynomial time?

Questions

Questions

Mechanisms

Round Robin

- Agents receive their most valued remaining good in a set order until no goods remain.

Envy-Cycle Elimination

- Let G be the envy graph on N
- $i \longrightarrow j$ in G iff $v_i(X_i) < v_i\left(X_j\right)$
- If a cycle is found, each agent can receive the bundle of the agent they envy in the cycle
- “Rotate” bundles around the cycle

Results

Computing EF1

Round Robin and Envy-Cycle Elimination produce an EF1 Allocation

Proof:

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Computing $\frac{1}{2}$ MMS

Round Robin and Envy-Cycle Elimination produce an EF1 Allocation

Proof:



Conclusion

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- Fair division is a wide field with various applications