
Fair Division Survey

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ABSTRACT

We survey common notions of fairness for allocating indivisible goods, their existence and computational guarantees, and open questions in fair division.

Keywords fair division · EFX · MMS · Nash welfare

1 Introduction

Fair division studies how resources can be allocated among agents with different preferences. A simple example is cut-and-choose: one agent divides a resource into two pieces of equal value in their eyes, and the other chooses a preferred piece. The procedure separates the ability to construct the division from the ability to choose, giving both agents a reason to accept the outcome.

Fair division has applications in estate division and food distribution, and its procedures have been studied for international dispute resolution [1], [2], [3], [4]. Steinhaus [5] helped formalize its mathematical study. Both divisible resources and indivisible goods have received considerable attention. For divisible cake, Aziz and Mackenzie [6] resolved the existence of a bounded envy-free protocol in the Robertson-Webb query model.

Indivisibility creates a structural obstacle: a single valuable good cannot be shared in a deterministic allocation. Exact envy-freeness therefore need not exist, and several weaker notions have been proposed. Some always exist but are difficult to compute; for others, even existence remains open. In this survey, we discuss common notions, their computational guarantees, and their relationships. We also give an elementary argument connecting maximum Nash welfare and equitability up to any positively valued good under identical additive valuations.

2 Preliminaries

Let $\mathcal{N}$ be a set of n agents and $\mathcal{M}$ a set of m goods. Agent $i \in \mathcal{N}$ has a valuation $v_i : 2^{\mathcal{M}} \rightarrow \mathbb{R}_{\geq 0}$, with $v_{i(\emptyset)} = 0$. Write $v_{i(g)}$ for $v_{i(\{g\})}$. Valuations are **additive** if $v_{i(S)} = \sum_{g \in S} v_{i(g)}$, and **monotone** if $v_{i(S)} \leq v_{i(T)}$ whenever $S \subseteq T \subseteq \mathcal{M}$.

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Unless stated otherwise, valuations are nonnegative and additive. Identical valuations satisfy $v_i = v$ for all agents, and binary additive valuations satisfy $v_{i(g)} \in \{0, 1\}$. We state other restrictions locally. Polynomial-time claims for additive valuations refer to rational or integer item values represented in binary. Pseudopolynomial time may depend on their numerical magnitudes. For general set functions, we use a value oracle and count bundle-value queries.

A complete allocation $X = (X_1, \dots, X_n)$ consists of pairwise disjoint bundles with union $\mathcal{M}$; bundles may be empty. Agent i 's utility is $v_{i(X_i)}$. A partial allocation may leave a pool $P = \mathcal{M} \setminus \bigcup_{i \in \mathcal{N}} X_i$ unallocated. Let $\Pi_{k(S)}$ denote all ordered partitions of S into k bundles, allowing empty bundles.

Equitability and Lorenz dominance compare utilities across agents and depend on their cardinal scales. Such comparisons require a specified normalization or common utility units. Envy comparisons evaluate both bundles using the same agent's valuation.

3 Envy

An allocation is **envy-free** (EF) if $v_{i(X_i)} \geq v_{i(X_j)}$ for all agents i, j . Otherwise, i envies j . Exact EF need not exist for indivisible goods.

Proof: Take two agents and one good valued positively by both. Whoever receives the empty bundle envies the other agent. ■

For explicitly represented additive valuations, deciding whether an EF allocation exists is NP-complete [7]. This motivates allowing hypothetical removal of goods to resolve envy.

3.1 EFX and zero-valued goods

An allocation is **envy-free up to any positively valued good** (EFX) if

$$v_{i(X_i)} \geq v_{i(X_j \setminus \{g\})} \quad \forall i, j \in \mathcal{N}, \forall g \in X_j \text{ with } v_{i(g)} > 0. \quad (1)$$

The stronger convention requiring the inequality for every $g \in X_j$, including zero-valued goods, is denoted EFX_0 . We use EFX for the positive-good convention throughout and explicitly identify stronger guarantees. The definitions coincide for strictly positive item values. Different papers use different conventions, so the distinction matters [8].

For two agents with additive valuations, an efficient cut-and-choose procedure is available. The cutter sorts goods by nonincreasing value to itself and repeatedly places the next good in a currently least-valued bundle. The chooser takes a preferred bundle.

Theorem

For two agents with nonnegative additive valuations, this procedure produces an EFX_0 allocation in polynomial time.

Proof: The chooser does not envy the cutter. For the cutter, consider a nonempty bundle B and its last added good h . Before adding h , $v(B \setminus \{h\})$ was at most the other bundle's value, which can only increase. Every $g \in B$ is worth at least h , so $v(B \setminus \{g\}) \leq v(B \setminus \{h\})$. Either bundle is therefore acceptable to the cutter up to removal of any good from the other bundle. ■

Finding a partition with the smallest possible utility difference would also give a useful cut-and-choose construction with suitable tie breaking, but that optimization is generally NP-hard. The descending-order method avoids it. The same method gives EFX_0 for any number of agents with identical additive valuations.

For three additive agents, Chaudhury et al. [9] give an EFX construction with pseudopolynomial time. A later work simplifies and improves it [10].

Open Problem 3.1.1: Do complete EFX allocations always exist for any number of agents with nonnegative additive valuations? Can they be computed in polynomial time?

Leximin maximizes the minimum utility, then the second-smallest utility, and so on. The leximin++ refinement also considers bundle size: sort agents by pairs $(v_{i(X_i)}, |X_i|)$, comparing utility first and size second, and lexicographically maximize the resulting list. Thus maximize the lowest utility, then its bundle size, then the next utility and its bundle size, and continue.

Theorem

Under identical nonnegative additive valuations, every leximin++ allocation is EFX_0 [11].

Proof: Suppose $v(X_i) < v(X_j \setminus \{g\})$. Move g from j to i . The recipient improves in utility when $v(g) > 0$, and otherwise in bundle size. The donor's remaining utility stays strictly above the recipient's old utility. In the sorted utility-size list, the earliest changed pair therefore improves: the donor cannot create a worse pair at or below the recipient's old utility. This contradicts maximality. ■

This proves existence, not polynomial-time optimization of leximin++. For identical additive valuations, the descending-order method already gives an efficient construction.

3.2 EF1

An allocation is **envy-free up to one good** (EF1) if, for every i, j , either $X_j = \emptyset$ or some $g \in X_j$ satisfies $v_{i(X_i)} \geq v_{i(X_j \setminus \{g\})}$. Thus removal of at most one good resolves envy. Budish [12] formally introduced this notion. For additive valuations, removing a good of maximum value to i from X_j suffices. For general monotone valuations, singleton value does not determine the effect of removal.

Round-robin repeatedly lets agents choose their favorite remaining good in a fixed order.

Round-Robin

Input: $(\mathcal{N}, \mathcal{M}, v)$ with nonnegative additive valuations

Output: An EF1 allocation X

```
1  $P \leftarrow \mathcal{M}; X_i \leftarrow \emptyset$  for every  $i; i \leftarrow 1$ 
2 while  $P \neq \emptyset$ 
3   Choose  $g \in \arg \max_{g \in P} v_i(g)$ 
4    $X_i \leftarrow X_i \cup \{g\}$ 
5    $P \leftarrow P \setminus \{g\}$ 
6    $i \leftarrow (i \bmod n) + 1$ 
7 return  $X$ 
```

Theorem

For nonnegative additive valuations, round-robin produces EF1 [4].

Proof: Fix i, j . If i picks before j , its k th pick is worth at least j 's k th pick to i , because that good was still available. Summing shows no envy. If i picks after j , its k th pick is worth at least j 's $(k + 1)$ st pick to i . Remove j 's first pick and sum to eliminate envy. Nonnegative values cover unmatched final picks. ■

Another EF1 procedure is **envy-cycle elimination** [7]. Its graph has an edge $i \rightarrow j$ when i envies j . An unenvied agent is a source, with indegree zero. If no source exists, there is a directed cycle. Rotate bundles so each participant receives the next participant's envied bundle, then recompute the graph.

Envy-Cycle Elimination

Input: $(\mathcal{N}, \mathcal{M}, v)$ with nonnegative additive valuations

Output: An EF1 and $\frac{1}{2}$ -EFX₀ allocation X

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1  $P \leftarrow \mathcal{M}; X_i \leftarrow \emptyset$  for every  $i$ 
2 while  $P \neq \emptyset$ 
3   while the envy graph has no source
4     Find a directed cycle  $(c_1, \dots, c_k, c_1)$ 
5     Simultaneously set  $X_{c_t} \leftarrow X_{c_{t+1}}$  using old bundles, with  $c_{k+1} = c_1$ 
6     Recompute the envy graph
7   Choose a source  $i$ , preferring agents with empty bundles
8   Choose  $g \in \arg \max_{g \in P} v_i(g)$ 
9    $X_i \leftarrow X_i \cup \{g\}; P \leftarrow P \setminus \{g\}$ 
10  Recompute the envy graph
11 return  $X$ 
```

Theorem

Envy-cycle elimination preserves EF1 for monotone valuations [7].

Proof: Initially all bundles are empty. Adding a good to an unenvied bundle creates only envy resolved by removing that good. A cycle rotation permutes existing bundles and increases each participant's own utility, leaving other agents' utilities fixed. Previously valid EF1 comparisons are preserved. Induction gives the guarantee. ■

For general monotone valuations, any remaining good can be allocated to a source. The favorite-good and empty-bundle choices in the displayed additive version give a stronger approximation.

3.3 Approximate EFX

For $\alpha \in (0, 1]$, an allocation is α -EFX if

$$v_{i(X_i)} \geq \alpha v_{i(X_j \setminus \{g\})} \quad \forall i, j, \forall g \in X_j \text{ with } v_{i(g)} > 0. \quad (2)$$

Requiring the inequality for every good gives α -EFX₀.

Open Problem 3.3.1: What is the largest constant α such that every nonnegative additive instance admits an α -EFX allocation? What guarantee can be achieved in polynomial time?

Plaut and Roughgarden [11] and Chan et al. [13] study half-EFX guarantees. Chan et al.'s algorithm uses matchings among unenvied agents. The following argument establishes the guarantee for the single-good version above.

Theorem

The displayed envy-cycle elimination algorithm produces $\frac{1}{2}$ -EFX₀ for nonnegative additive valuations.

Proof: Own utility never decreases. While a bundle is empty, an empty-bundle agent is a source, so every agent receives a first favorite good before any bundle receives its second good. Fix i and a final bundle B of size at least two, tracking it through rotations. Let h be its last added good. When i received its first good, h was still unallocated; final utility is therefore at least $v_{i(h)}$. Immediately before h was added, B was unenvied, so i 's then-current utility was at least $v_{i(B \setminus \{h\})}$. Hence $2v_{i(X_i)} \geq v_{i(B)} \geq v_{i(B \setminus \{g\})}$ for every $g \in B$. Empty and singleton bundles are immediate. ■

Amanatidis et al. [14] improve the general additive guarantee to $\varphi - 1 \approx 0.618$, where $\varphi = \frac{1+\sqrt{5}}{2}$. Their 2024 work [15] gives $\frac{2}{3}$ -EFX in restricted additive settings: at most seven agents, at most three distinct item values, or multigraph valuations. It does not establish $\frac{2}{3}$ -EFX for unrestricted additive or general monotone valuations.

4 Pareto Optimality

Allocation A **Pareto dominates** B if $v_{i(A_i)} \geq v_{i(B_i)}$ for all agents, strictly for at least one. An allocation is **Pareto optimal** (PO) if no feasible allocation dominates it. Weak coordinatewise dominance is a partial order on utility vectors, although distinct allocations may have the same vector.

Fairness and efficiency can conflict. The following example concerns the zero-inclusive convention.

Theorem

Some additive instance admits no allocation that is both PO and EFX_0 [11].

Proof: Consider two agents with item values:

Agent	a	b	c
1	2	1	0
2	2	0	1

PO requires giving b to agent 1 and c to agent 2. If a goes to agent 1, removing b leaves agent 2 facing value 2 while its own value is 1. If a goes to agent 2, the symmetric violation occurs. Thus every PO allocation fails EFX_0 . This does not establish incompatibility with positive-good EFX. ■

Lorenz dominance compares utility vectors [16]. Sort both vectors in nondecreasing order. Vector a weakly Lorenz dominates b if $\sum_{i=1}^k a_i \geq \sum_{i=1}^k b_i$ for every $k \in \{1, \dots, n\}$. Strict Lorenz dominance requires at least one strict inequality. The comparison depends on interpersonal utility scales.

5 MMS

Agent i 's **maximin share** (MMS) is the greatest minimum bundle value it can ensure by partitioning the goods into n bundles:

$$\mu_i^{n(\mathcal{M})} = \max_{B \in \Pi_{n(\mathcal{M})}} \min_{k \in \{1, \dots, n\}} v_{i(B_k)}. \quad (3)$$

An MMS allocation satisfies $v_{i(X_i)} \geq \mu_i^{n(\mathcal{M})}$ for all agents. Each benchmark uses that agent's valuation alone; this differs from globally maximizing the worst-off agent's utility.

The benchmark always exists for a finite instance, but an allocation meeting all benchmarks need not exist [17]. Computing exact MMS is NP-hard. High-probability results require a specified random-valuation model. An α -MMS allocation satisfies $v_{i(X_i)} \geq \alpha \mu_i^{n(\mathcal{M})}$, where $\alpha \in (0, 1]$.

5.1 A half-MMS construction

For an existence argument, normalize each positive MMS benchmark to 1. Zero-benchmark agents impose no positive requirement and can be set aside; reducing the active-agent count cannot lower other agents' benchmarks. Exact normalization is conceptual, not an efficient preprocessing step.

First give any good worth at least $\frac{1}{2}$ to an unserved agent who values it that much. Then fill bags with arbitrary remaining goods until an unserved agent values the bag at least $\frac{1}{2}$. Allocate that bag and continue. The last agent receives the remaining goods.

Bag-Filling with Normalized Positive MMS Benchmarks

Input: An additive instance with $\mu_i^{n(\mathcal{M})} = 1$
Output: A $\frac{1}{2}$ -MMS allocation X

- 1 $R \leftarrow \mathcal{N}; P \leftarrow \mathcal{M}; X_i \leftarrow \emptyset$ for every i
- 2 **while** some $i \in R, g \in P$ satisfy $v_{i(g)} \geq \frac{1}{2}$
- 3 | $X_i \leftarrow \{g\}; R \leftarrow R \setminus \{i\}; P \leftarrow P \setminus \{g\}$
- 4 **if** $R = \emptyset$, distribute leftover goods arbitrarily and **return** X
- 5 **while** $|R| > 1$
- 6 | $B \leftarrow \emptyset$
- 7 | **while** $v_{i(B)} < \frac{1}{2}$ for every $i \in R$
- 8 | | Choose $g \in P; P \leftarrow P \setminus \{g\}; B \leftarrow B \cup \{g\}$
- 9 | Choose $i \in R$ with $v_{i(B)} \geq \frac{1}{2}$
- 10 | $X_i \leftarrow B; R \leftarrow R \setminus \{i\}$
- 11 Give P to the remaining agent and **return** X

Theorem

Removing one agent and one good weakly increases remaining agents' MMS benchmarks when the number of bundles is reduced [18]. For $n \geq 2$,

$$\mu_i^{n-1}(\mathcal{M} \setminus \{g\}) \geq \mu_i^{n(\mathcal{M})}. \quad (4)$$

Proof: Fix remaining agent i and a partition attaining its n -bundle benchmark. Discard the bundle containing g , distributing its other goods among the remaining $n - 1$ bundles. Nonnegativity preserves each remaining bundle's value to i . This does not assume a simultaneous MMS allocation exists. ■

Theorem

The normalized construction produces $\frac{1}{2}$ -MMS [18].

Proof: Singleton reductions preserve a benchmark of at least 1 for each unserved agent. With r agents remaining after this phase, each values the pool at least r . Each remaining good is worth less than $\frac{1}{2}$ to every remaining agent.

Before a bag's last addition, all unserved agents value it below $\frac{1}{2}$. The last good is also below $\frac{1}{2}$, so the completed bag is worth less than 1 to every agent left unserved. Removing a recipient and a bag preserves the invariant that each remaining agent values the pool at least the number of remaining agents. Bags can reach the threshold, every recipient gets at least $\frac{1}{2}$, and the final agent gets at least 1. Allocating leftovers cannot harm these guarantees. ■

A polynomial-time algorithm must handle unknown benchmarks using suitable estimates or the cited approximation methods. The normalized listing establishes the guarantee, not the running time.

5.2 Approximation bounds

Procaccia and Wang [19] establish $\frac{2}{3}$ -MMS with efficient computation for a fixed number of agents. Later algorithms handle arbitrary agent counts, including Garg et al. [18]. Garg and Taki [20] give $\frac{3}{4}$, and Akrami et al. [21] obtain

$$\frac{3}{4} + \min\left\{\frac{1}{36}, \frac{3}{16n-4}\right\}. \quad (5)$$

The additional term tends to zero with n . Akrami and Garg [22] obtain $\frac{3}{4} + \frac{3}{3836}$, a fixed improvement over $\frac{3}{4}$.

Feige et al. [23] give a three-agent instance ruling out a uniform guarantee greater than $\frac{39}{40}$. This is not a known tight answer.

Open Problem 5.2.1: What is the largest constant α guaranteeing α -MMS for every nonnegative additive instance? Can the known lower and upper bounds be improved?

6 Maximum Nash Welfare

Maximum Nash welfare (MNW) considers agents' own utilities together. When all utilities can be positive, maximizing their product is equivalent to maximizing their geometric mean. If some must be zero, the product over all agents would always be zero. We therefore use a support-first definition [4].

Let $N^+(X) = \{i \in \mathcal{N} : v_{i(X_i)} > 0\}$. Lexicographically maximize

$$\left(|N^+(X)|, \prod_{i \in N^+(X)} v_{i(X_i)} \right) \quad \text{over } X \in \Pi_n(\mathcal{M}), \quad (6)$$

with an empty product equal to 1. First maximize positive support, then the product of positive utilities. A product rewards balancing when transfers are feasible, but this intuition does not characterize the indivisible optimum.

Theorem

Every MNW allocation is PO and EF1 for nonnegative additive valuations [4].

Proof: A Pareto improvement increases positive support or preserves it and increases the positive utility product. Thus MNW implies PO.

Suppose EF1 fails for i, j : $v_{i(X_i)} < v_{i(X_j \setminus \{g\})}$ for every $g \in X_j$. Put $A = v_{i(X_i)}$, $B = v_{j(X_j)}$, and $C = v_{i(X_j)}$. If $A, B > 0$, choose g with $s = v_{i(g)} > 0$ minimizing $\frac{t}{s}$, where $t = v_{j(g)}$. Weighted averaging over goods positively valued by i gives $\frac{t}{s} \leq \frac{B}{C}$. Failure of EF1 gives $C > A + s$, hence

$$\frac{t}{B} \leq \frac{s}{C} < \frac{s}{A + s}. \quad (7)$$

Moving g gives $(A + s)(B - t) > AB$ with both utilities positive, contradicting MNW.

If $A = 0$ and $B > 0$, failure of EF1 requires at least two goods liked by i in X_j . One can be moved while leaving j positive, increasing support. If $A > 0$ and $B = 0$, moving a liked good increases i 's utility without changing j 's. If both are zero, moving a liked good increases support. All cases contradict MNW. ■

MNW also gives $\frac{2}{1+\sqrt{4n-3}}$ -MMS [4], but need not give EFX for general additive valuations [8].

Exact MNW is NP-hard even for identical additive valuations [24]. Binary additive MNW is polynomial-time computable [25]. Positive two-value instances require a finer distinction: normalize the values to $\{1, \frac{p}{q}\}$ for coprime integers $p > q \geq 1$. Polynomial-time algorithms are known for $q = 1, 2$, while computing an optimum is NP-hard for $q \geq 3$ [26]. Arbitrary bivalued instances are therefore not all tractable. Three-value instances can also be NP-hard [8].

These barriers motivate computing fairness and efficiency directly. Barman et al. [27] obtain EF1 and PO in pseudopolynomial time for integer-valued additive instances, and polynomial time when item values are polynomially bounded.

Open Problem 6.1: Can EF1 and PO be computed in polynomial time for general nonnegative additive valuations?

Exact MNW always exists in a finite instance. The meaningful approximation question concerns efficient computation. For instances admitting positive utility for all agents, an α approximation to Nash social welfare achieves at least α times the optimal geometric mean. A factor for the product differs from a factor for its n th root. Welfare approximation alone need not preserve EF1 or PO.

Open Problem 6.2: What is the best polynomial-time approximation guarantee for Nash social welfare under additive valuations?

7 EQX

An allocation is **equitable** (EQ) if $v_{i(X_i)} = v_{j(X_j)}$ for all i, j , on specified cardinal scales. Equality may be impossible with indivisible goods. **Equitability up to any positively valued good** (EQX) requires

$$v_{i(X_i)} \geq v_{j(X_j \setminus \{g\})} \quad \forall i, j, \forall g \in X_j \text{ with } v_{j(g)} > 0. \quad (8)$$

The positivity test uses the owner's valuation v_j , whereas EFX uses the comparing agent's v_i . Under identical valuations the notions coincide with matching conventions. Freeman et al. [28] show that strictly positive additive goods admit EQX and PO together, and give a pseudopolynomial EQ1 and PO algorithm.

Barman et al. [29] extend EQX existence to monotone valuations. Exact polynomial-time computation is available for weakly well-layered valuations, including additive valuations; their general monotone efficient guarantee is approximate EQX. Chores use the corresponding nonincreasing valuation model, and mixed manna requires further assumptions.

Theorem

Under identical nonnegative additive valuations, every MNW allocation is EQX with the positive-good convention.

Proof: Let $a = v(X_i)$, $b = v(X_j)$, $c = v(g) > 0$, and suppose $a - c > b$. Move g from i to j . If $b > 0$, both remain positive and

$$(a - c)(b + c) - ab = c(a - b - c) > 0. \quad (9)$$

This increases the positive utility product. If $b = 0$, the donor remains positive and the recipient becomes positive, increasing support. Either contradicts MNW. ■

Positivity is essential for arbitrary MNW tie breaking. With identical item values 2, 1, 0, allocation $(\{2, 0\}, \{1\})$ is MNW, but removing the zero-valued good leaves value 2 facing value 1. Thus arbitrary MNW does not guarantee the stronger all-goods EQX analogue. Strictly positive item values remove this distinction.

8 Between EFX and EF1

Envy-freeness up to a less preferred good (EFL) lies between positive-good EFX and EF1 for additive valuations [30]. For each i, j , either X_j contains at most one good positively valued by i , or there exists $g \in X_j$ such that

$$v_{i(X_i)} \geq \max\{v_{i(X_j \setminus \{g\})}, v_{i(g)}\}. \quad (10)$$

Both comparisons use i 's valuation. The removed good must resolve envy and itself be worth no more to i than its own bundle. EFL can be computed in polynomial time using a suitable envy-cycle procedure.

8.1 Envy-freeness up to a random good

Farhadi et al. [31] introduce **envy-freeness up to a random good** (EFR). For nonempty X_j , let G be uniformly distributed over its goods. EFR requires

$$v_{i(X_i)} \geq \mathbb{E}[v_{i(X_j \setminus \{G\})}]. \quad (11)$$

For additive valuations, the right side is $\left(1 - \frac{1}{|X_j|}\right)v_{i(X_j)}$; empty-bundle comparisons are immediate. This is a hypothetical deletion from a fixed allocation, not necessarily a randomized allocation mechanism. An α -EFR allocation multiplies the right side by α . The cited work gives a polynomial-time 0.73-EFR algorithm.

Strategyproofness of randomized mechanisms is separate. An input-independent uniform allocation is strategyproof and ex ante envy-free but may be inefficient. Impossibility results for strategyproof random assignment include additional requirements, rather than ruling out these two properties alone [32].

9 Improving the Maximin Share

9.1 GMMS

Groupwise maximin share (GMMS) requires MMS fairness within every coalition using only its allocated goods [30]:

$$v_{i(X_i)} \geq \mu_i^{|S|}(\cup_{j \in S} X_j) \quad \forall S \subseteq \mathcal{N}, i \in S. \quad (12)$$

Let $\text{GMMS}_{i(X)}$ be the maximum of these benchmarks over coalitions containing i . An α -GMMS allocation satisfies $v_{i(X_i)} \geq \alpha \cdot \text{GMMS}_{i(X)}$. The full coalition recovers MMS, so GMMS need not exist.

Barman et al. [30] give a $\frac{1}{2}$ guarantee, Amanatidis et al. [14] improve it to $\frac{2}{\varphi+2}$, and Chaudhury et al. [33] give $\frac{4}{7}$ -GMMS for complete additive allocations. Existence guarantees and construction running times must be distinguished.

9.2 PMMS

Pairwise maximin share (PMMS) restricts GMMS to pairs:

$$v_{i(X_i)} \geq \mu_i^2(X_i \cup X_j) \quad \forall i \neq j. \quad (13)$$

For α -PMMS, multiply each pairwise benchmark by α . A known general additive approximation is approximately 0.781 [34].

Open Problem 9.2.1: What are the best uniform approximation guarantees for GMMS and PMMS under additive valuations?

For nonnegative additive valuations the hierarchy is

$$\text{EF} \Rightarrow \text{GMMS} \Rightarrow \text{PMMS} \Rightarrow \text{EFX} \Rightarrow \text{EFL} \Rightarrow \text{EF1}, \quad (14)$$

and GMMS also implies MMS [30]. EFX here uses positive-good removal. The implication to EFX_0 fails: the identical-values allocation $(\{2, 0\}, \{1\})$ has two-way share 1, so is GMMS and PMMS, but is not EFX_0 .

Theorem

An allocation can satisfy both MMS and PMMS while failing GMMS.

Proof: Take nine agents and eight goods. Agent 1's values are $(5, 5, 5, 3, 3, 1, 1, 1)$. Agents 2, 3, and 4 value every good at 1; agents 5 through 9 value every good at 0. Allocate

$$X_1 = \{g_1\}, \quad X_2 = \{g_2, g_4\}, \quad X_3 = \{g_3, g_5\}, \quad X_4 = \{g_6, g_7, g_8\}, \quad (15)$$

with the other bundles empty. All nine-bundle MMS benchmarks are zero, so MMS holds.

Zero-valued agents satisfy PMMS automatically. For $i \in \{2, 3, 4\}$, the pairwise benchmark is $\left\lfloor \frac{|X_i| + |X_j|}{2} \right\rfloor$, no larger than its own bundle size. For agent 1, pairing with agents 2 or 3 gives total value 13 and two-way share 5; pairing with agent 4 gives total value 8 and share at most 4; pairing with an empty bundle gives share 0. Thus PMMS holds.

Within coalition $\{1, 2, 3, 4\}$, agent 1 can partition all goods into values $5 + 1$, $5 + 1$, $5 + 1$, and $3 + 3$. Its four-way benchmark is at least 6, above its utility 5, so GMMS fails. This example adapts the construction in [30]. ■

10 Other EFX Notions

10.1 Epistemic EFX

Epistemic envy-freeness (EEF) asks whether each agent i , holding its bundle fixed, can redistribute $\mathcal{M} \setminus X_i$ among the others to eliminate its envy [35]. Epistemic EFX (EEFX) requires that imagined redistribution to satisfy EFX for i [36]. Different agents may use different imagined redistributions. In the general monotone results below, EFX satisfaction uses every single-good removal, rather than testing singleton values.

Caragiannis et al. [36] prove existence and polynomial-time computation for additive valuations. Akrami and Rathi [37] extend existence to general monotone valuations. This does not give general polynomial-time computation: EEFX is PLS-hard and requires exponentially many value queries even for identical submodular valuations.

EEF need not exist. With two agents and one good valued positively by both, the empty-bundle agent has only one possible recipient for the remaining good and cannot remove its envy by redistribution.

10.2 Charity

One can relax completeness by leaving a pool P unallocated while requiring EFX among allocated bundles. The additional **no-envy-of-charity** requirement is $v_{i(X_i)} \geq v_{i(P)}$ for all agents. Leaving every good unallocated satisfies bundle comparisons vacuously but generally fails this requirement.

Chaudhury et al. [33] give a pseudopolynomial algorithm leaving at most $n - 1$ goods unallocated while no agent envies the pool. Berger et al. [38] obtain partial EFX leaving at most

$n - 2$ goods for $n \geq 3$, and at most one good for four agents. These latter bounds should not automatically be interpreted as inheriting every additional property of the earlier construction.

Open Problem 10.2.1: Can every additive instance admit partial EFX with a sublinear number of unallocated goods? What is possible if no agent may envy the pool?

Zero unallocated goods recovers complete EFX. A related relaxation is **EFkX**: for every i, j , remove any $T \subseteq X_j$ of size $\min\{k, |X_j|\}$ and require $v_{i(X_i)} \geq v_{i(X_j \setminus T)}$. We use zero-inclusive removal here, so EF1X coincides with EFX₀. Akrami et al. [39] establish EF2X for restricted additive valuations, where $v_{i(g)} \in \{0, v(g)\}$.

Arbitrary completion of partial EFX does not automatically give EF2X. Take three agents with identical values, seven unit goods, and a good h worth 2. Give the agents two, three, and two unit goods, leaving h unallocated. This is EFX₀ and no agent envies the pool. Giving h to the three-good bundle leaves value 3 after removing two unit goods, above either other agent's utility 2. A valid completion guarantee needs a separate argument.

11 Different Settings

11.1 Online Fair Division

Offline division has all goods available in advance. Online models allow agents, goods, or both to arrive over time [40]. Here we focus on irrevocably allocating each arriving good without knowing future goods or valuations.

Benade et al. [41] consider additive item values in $[0, 1]$. Let $\text{Envy}_T = \max_{i,j} \{v_{i(X_j)} - v_{i(X_i)}\}$ after T arrivals. Uniform random assignment gives $\mathbb{E}[\text{Envy}_T] = O\left(\sqrt{T \frac{\log(T)}{n}}\right)$ when $T \geq n \log(T)$, even against an adaptive adversary. Expected envy per arrival therefore vanishes for fixed n . Zero expected signed envy of a fixed pair is different from zero expected maximum envy. The paper also develops a deterministic method and lower bounds matching up to logarithmic factors.

Ex ante EF compares expected utilities. Ex ante PO rules out a lottery improving every agent's expected utility weakly and one strictly. Ex post PO requires every allocation realized with positive probability to be PO.

Theorem

With general nonnegative additive cardinal utilities, no online mechanism guarantees ex ante EF and ex ante PO on every sequence [42].

Proof: Consider:

Agent	g_1	g_2
1	1	2
2	2	1

The sequence might end after g_1 , so ex ante EF forces a half-half split of that good. Irrevocability retains these probabilities when g_2 arrives. If q is its probability of going to agent 1, agent 1's envy inequality is $\frac{1}{2} + 2q \geq \frac{1}{2} + 2(1 - q)$, giving $q \geq \frac{1}{2}$. Agent 2's is $2 - q \geq 1 + q$, giving $q \leq \frac{1}{2}$. Expected utilities are $(\frac{3}{2}, \frac{3}{2})$, Pareto dominated by giving each its favorite good, yielding $(2, 2)$. The obstruction is a universal online guarantee; this two-good instance itself admits an EF and PO allocation. ■

11.2 Chores and Mixed Manna

For chores, use nonnegative costs c_i , with lower costs preferred. For chores we use the zero-inclusive removal convention. Chores-EFX requires

$$c_{i(X_i \setminus \{g\})} \leq c_{i(X_j)} \quad \forall i, j, \forall g \in X_i. \quad (16)$$

Removal is from the envying agent's own bundle. A multiplicative cost approximation uses $\beta \geq 1$ on the right side.

Christoforidis and Santorinaios [43] give a superadditive-cost instance with no EFX allocation and an unbounded approximation ratio, and establish NP-completeness of deciding existence for three agents with suitably represented superadditive costs. Their removal convention includes zero-cost chores, which matters for the construction. These results do not settle additive chores-EFX existence, which remained open in December 2024, or make fairness and PO incompatible in every instance.

Restricted domains allow stronger guarantees. Ebadian et al. [44] study bivalued additive chores, while Barman et al. [45] study supermodular costs with binary marginal values. These are different classes; their EF1 and PO guarantees must be read with those restrictions. Competitive chore division has its own equilibrium algorithms [46].

Mixed manna permits $v_i : 2^{\mathcal{M}} \rightarrow \mathbb{R}$: an item can be a good for one agent and a chore for another. Barman et al. [29] show EQX nonexistence even for two agents with additive normalized valuations. Deciding existence is weakly NP-hard for two agents and strongly NP-hard for unrestricted agent counts. When every item has the same sign for all agents, some stronger guarantees are available.

11.3 Incentives

A mechanism is **strategyproof** if an agent cannot improve its utility by misreporting its valuation while other reports stay fixed. For randomized mechanisms the comparison usually uses expected utility. This is a property of the allocation rule, not of a single allocation.

Fairness and truthfulness can conflict. Amanatidis et al. [47] show that for two agents and at least four goods, no deterministic truthful mechanism without payments guarantees α -MMS for $\alpha > \frac{1}{2}$ on general additive valuations. Such impossibilities concern a conjunction of properties across all inputs; they do not deny truthful mechanisms or fair allocations for a fixed instance. Related characterization results identify achievable guarantees [48].

For binary additive valuations, Halpern et al. [49] construct a group-strategyproof MNW mechanism with lexicographic tie breaking. Arbitrary selection of MNW optima is not enough to conclude truthfulness. Binary additive EF1 implies positive-good EFX because all positively valued goods have equal value. Babaioff et al. [16] study truthful fair mechanisms for dichotomous submodular valuations, whose marginal values are binary, rather than arbitrary submodular valuations.

12 Proportionality

An allocation is **proportional** (PROP) if $v_{i(X_i)} \geq \frac{v_{i(\mathcal{M})}}{n}$ for every agent. Indivisibility can prevent PROP. **Proportionality up to one good** (PROP1) permits adding at most one outside good: some $T \subseteq \mathcal{M} \setminus X_i$ with $|T| \leq 1$ satisfies $v_{i(X_i \cup T)} \geq \frac{v_{i(\mathcal{M})}}{n}$.

Proportionality up to any positively valued good (PROPx) requires either that the agent is already proportional or that adding every outside good g with $v_{i(g)} > 0$ reaches $\frac{v_{i(\mathcal{M})}}{n}$. The zero-value convention should be specified when comparing variants. Aziz, Moulin, and Sandomirskiy [50] give a strongly polynomial PROP1 and PO algorithm for additive valuations.

Theorem

Identical strictly positive additive valuations need not admit PROPx [50].

Proof: Take three agents and five goods valued 3, 3, 3, 3, 1 by all agents. The target is $\frac{13}{3}$. The two agents not holding the unit good must each have utility at least 6: if not already proportional, adding that outside unit good must reach $\frac{13}{3}$, and their utilities are multiples of 3. They consume all four value-3 goods, leaving only the unit good for its holder. Adding a value-3 good then gives $4 < \frac{13}{3}$, a contradiction. ■

Proportionality up to the maximin good (PROPM) is an intermediate notion that always exists and is polynomial-time computable for additive goods [51]. Define

$$d_{i(X)} = \max_{j \neq i} \min_{g \in X_j} v_{i(g)}, \quad (17)$$

taking the minimum of an empty bundle as 0 and $d_{i(X)} = 0$ for one agent. PROPM requires $v_{i(X_i)} + d_{i(X)} \geq \frac{v_{i(\mathcal{M})}}{n}$. The threshold is $\frac{1}{n}$ only after normalizing $v_{i(\mathcal{M})} = 1$.

13 Fisher Markets

Markets offer another route to fairness and efficiency. A linear Fisher market has divisible goods, positive budgets ℓ_i , and prices $p = (p_1, \dots, p_m)$. A fractional allocation gives utility

$$u_{i(X_i)} = \sum_{g \in \mathcal{M}} v_{i(g)} x_{i,g}, \quad (18)$$

where $x_{i,g} \geq 0$ and $\sum_i x_{i,g} \leq 1$. The feasible set differs from integral allocation of indivisible goods.

Assume every agent values some good and every good is valued by someone. An equilibrium satisfies:

1. Supply clears: $\sum_i x_{i,g} = 1$ for each g .
2. Budgets are exhausted: $\sum_g p_g x_{i,g} = \ell_i$ for each i .
3. Agents receive only maximum-bang-per-buck goods: $x_{i,g} > 0$ implies $\frac{v_{i(g)}}{p_g} = \max_h \frac{v_{i(h)}}{p_h}$.

These assumptions permit positive prices; goods valued by nobody can be assigned price zero and handled separately. Equal budgets give a **competitive equilibrium from equal incomes** (CEEI).

Exact equilibria for rational linear divisible-goods Fisher markets are polynomial-time computable, including by the classical combinatorial algorithm of Devanur et al. [52]. Exact CEEI computation is therefore not open for this model. Integral CEEI need not exist: one positive-priced indivisible good cannot exhaust two equal positive budgets in an integral allocation.

An integral allocation is **fractionally Pareto optimal** (fPO) if no feasible fractional allocation dominates it. Thus fPO implies PO, since integral allocations form a subset of the fractional feasible set. A divisible-goods CEEI is envy-free and efficient within the fractional feasible set under the standard linear assumptions [53]. Producing integral fairness from an equilibrium needs an additional rounding or allocation argument.

Chore markets differ: agents earn payments for taking bids rather than spend budgets on goods. The approximate-equilibrium algorithms of Boodaghians et al. [54] and Chaudhury et al. [55] concern chores. Their definitions and guarantees cannot be transferred directly to the divisible-goods model above.

14 Empirical Results

Formal guarantees address stated valuations, while perceived fairness may also depend on agents' information. Hosseini et al. [56] use crowdsourced scenarios to compare preferences for assigned and alternative bundles under different fairness and information conditions. The information presented can affect perceived fairness. This differs from a universal guarantee that agents accept an EF1 allocation.

Cardinal elicitation is another practical issue. Spliddit asks agents to distribute 1,000 points among goods according to their preferences. Positive rescaling preserves envy comparisons. When every agent can receive positive utility, rescaling also preserves MNW optima; interpersonal notions such as EQX depend on the normalization. Caragiannis et al. [4] report sub-three-second MNW computation on each of 1,281 Spliddit instances, whose largest instances had ten agents, with the reported hardware and solver. This is an experimental benchmark, not a worst-case runtime guarantee for all ten-agent inputs.

15 Discussion

We surveyed common fairness notions for indivisible goods and their connections to efficiency and computation. Indivisibility prevents exact EF and PROP even in simple instances, motivating EF1, EFX, and MMS approximations. Stronger guarantees may require restricted valuations, unallocated goods, or a different computational objective.

MNW illustrates both the usefulness and difficulty of a global objective: it guarantees EF1 and PO under additive valuations but may be hard to compute exactly. With identical additive valuations, it also gives EQX for positively valued goods by the elementary transfer argument above. The qualification about zeros is necessary for stronger removal conventions.

As of December 2024, central questions include complete additive EFX existence, polynomial-time EF1 and PO computation, and tighter MMS and groupwise approximation guarantees. Strategic and online models add different obstacles; their impossibilities concern mechanisms with specified information and incentive requirements. Keeping these models distinct is essential when comparing what is known and what remains unproved.

16 References

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